Simultaneous deformations of algebras and morphisms via derived brackets

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Abstract

We present a method to construct explicitly L_∞-algebras governing simultaneous deformations of various kinds of algebraic structures and of their morphisms. It is an alternative to the heavy use of the operad machinery of the existing approaches. Our method relies on Voronov's derived bracket construction.

Contents

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Introduction

The deformation theory of various kinds of structures (e.g. [17], [18], [19] and [20], [10]) can be encapsulated in the language of graded Lie algebras ([24] and [25]) or more generally, for non quadratic structures, of L_∞-algebras ([21]).

It is convenient to have such a formulation since cohomology theory, analogues of Massey products and a natural equivalence relation on the space of deformations come along for free. However obtaining such formulation – that is, obtaining the L_∞- algebra governing the given deformation problem – can be difficult.

There are known techniques ([7] and [23]) to solve this problem in the case of simultaneous deformations of various kinds of algebras and their morphisms, but they are based on the formalism of operads, which provide an obstacle to mathematicians not acquainted with operad theory.

On the other hand, T. Voronov developed techniques enabling to produce L_∞-algebras out of some simple concepts of graded linear algebra ([33] and [34]). In our work [8] we showed how to adapt Voronov's results to the study of simultaneous deformations, and gave geometrical applications which could not be obtained otherwise.

In this paper we show that this approach also applies successfully to simultaneous deformations of algebras and morphisms and that this can be an alternative approach for users not willing to use the operadic formalism.

Outline of the content of the paper.

In §1 we recall the formalism of graded Lie algebras and L_∞-algebras together with the *derived bracket constructions* (see Thm. 1 and 2) and the tool we use to study simultaneous deformations (Thm. 3).

We give algebraic applications to the study of simultaneous deformations of algebras and morphisms in the following categories: Lie, L_∞, Lie bialgebras (see §2) and associative algebras (see §3). Another application concerns Lie subalgebras of Lie algebras. More generally, we show in §4 how this works in general for algebras and morphisms over any Koszul operad.

In the appendix we provide background material on graded and formal geometry.

Acknowledgements: We thank J. Stasheff for comments, and D. Iacono, M. Manetti, F. Schätz, B. Shoiket, B. Vallette, T. Willwacher for useful conversations.
M.Z. thanks Uni.lu for hospitality (Luxembourg, 08/2010, grant FNR/10/AM2c/18). He was partially supported by CMUP (Porto), financed by FCT (programs POCTI, POSI and Ciencia 2007); grants PTDC/MAT/098770/2008 and PTDC/MAT/099880/2008 (Portugal), MICINN RYC-2009-04065 and MTM2009-08166-E (Spain).

Most of the work of Y.F. on this article was done while assistant of Prof. Dr. Martin Schlichenmaier at Uni.lu (grant R1F105L15), to whom he would like to address his warmest thanks. He benefited form the support of the UAM through grant MTM2008-02686 (Madrid, 06/2010), and from the MPIM in Bonn (12/2011-01/2012).

1 L_∞-algebras via derived brackets and Maurer-Cartan elements

We recall the machinery we developed in [8, §1] (which first appeared as [9, §1]). The main result is Thm. 3, which produces the L_∞-algebras appearing in the rest of the article. We first give some basic material about L_∞-algebras in §1.1, then we recall in §1.2 Voronov's constructions which will be the main tools used to establish in §1.3 our Theorem 3. We conclude justifying in §1.4 why no convergence issues arise in our machinery, and discussing equivalences in §1.5.

We refer the reader to [8, §1] for additional details and proofs (an exception being Lemma 1.13, which we prove here).

1.1 Background on L_∞-algebras

The notion of L_∞-algebra is due to Lada and Stasheff [21], and contains graded Lie algebras and differential graded Lie algebras (DGLAs) as special cases. We will need only a "shifted" version of this notion, in which all the multibrackets are graded symmetric have degree one. We refer to the latter as $L_\infty[1]$-algebras.

To introduce it, recall that given two elements v_1, v_2 in a graded vector space, the *Koszul sign* of the transposition $\tau_{1,2}$ of these two elements is $\epsilon(\tau_{1,2}, v_1, v_2) := (-1)^{|v_1||v_2|}$. This definition is extended to an arbitrary permutation using a its decomposition into transpositions.

Recall further that $\sigma \in S_n$ is called an $(i, n-i)$-unshuffle if it satisfies $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. The set of $(i, n-i)$-unshuffles is denoted by $S_{(i,n-i)}$.

Definition 1.1. ([16, Def. 5]) An $L_\infty[1]$-*algebra* is a graded vector space $W = \bigoplus_{i\in\mathbb{Z}} W_i$ equipped with a collection ($k \geq 1$) of linear maps $m_k \colon \otimes^k W \longrightarrow W$ of degree 1 satisfying, for every collection of homogeneous elements $v_1, \ldots, v_n \in W$:

1) graded symmetry: for every $\sigma \in S_n$

$$m_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \epsilon(\sigma) m_n(v_1, \ldots, v_n),$$

2) relations: for all $n \geq 1$

$$\sum_{\substack{i+j=n+1\\ i,j\geq 1}} \sum_{\sigma \in S_{(i,n-i)}} \epsilon(\sigma) m_j(m_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0.$$

In a *curved* $L_\infty[1]$-*algebra* one additionally allows for an element $m_0 \in W_1$ (which can be understood as a bracket with zero arguments), one allows i and j to be zero in the relations 2), and one adds the relation corresponding to $n = 0$.

Remark 1.2. There is a bijection between L_∞-algebra structures on a graded vector space V and $L_\infty[1]$-algebra structures on $V[1]$, the graded vector space defined by $(V[1])_i := V_{i+1}$ [33, Rem. 2.1]. The multibrackets are related by applying the décalage isomorphisms

$$(\otimes^n V)[n] \cong \otimes^n(V[1]), \quad v_1 \dots v_n \mapsto v_1 \dots v_n \cdot (-1)^{(n-1)|v_1|+\dots+2|v_{n-2}|+|v_{n-1}|}, \tag{1}$$

where $|v_i|$ denotes the degree of $v_i \in V$. The bijection extends to the curved case.

From now on, for any $v \in V$, we denote by $v[1]$ the corresponding element in $V[1]$ (which has degree $|v|-1$). Also, we denote the multibrackets in $L_\infty[1]$-algebras by $\{\cdots\}$, we denote by $d := m_1$ the unary bracket, and in the curved case we denote $\{\emptyset\} := m_0$ (the bracket with zero arguments).

Definition 1.3. Given an $L_\infty[1]$-algebra W, a *Maurer-Cartan element* is a degree 0 element Φ satisfying the Maurer-Cartan equation

$$\sum_{n=1}^{\infty} \frac{1}{n!}\{\underbrace{\Phi,\dots,\Phi}_{n \text{ times}}\} = 0. \tag{2}$$

(We consider the convergence of this infinite sum in §1.4). We denote by $MC(W)$ the set of Maurer-Cartan elements of W.

If W is a curved $L_\infty[1]$-algebra, we define Maurer-Cartan elements by adding $m_0 \in W_1$ to the left hand side of eq. (2) (i.e. by letting the sum in (2) start at $n=0$).

1.2 Th. Voronov's constructions of L_∞-algebras as derived brackets

We recall Th. Voronov's derived bracket construction [33][34], which out of simple data constructs an $L_\infty[1]$-algebra structure.

Definition 1.4. A *V-data* consists of a quadruple $(L, \mathfrak{a}, P, \Delta)$ where

- L is a graded Lie algebra (we denote its bracket by $[\cdot,\cdot]$),
- $\mathfrak{a}$ an abelian Lie subalgebra,
- $P\colon L \to \mathfrak{a}$ a projection whose kernel is a Lie subalgebra of L,
- $\Delta \in Ker(P)_1$ an element such that $[\Delta,\Delta]=0$.

When Δ is an arbitrary element of L_1 instead of $Ker(P)_1$, we refer to $(L, \mathfrak{a}, P, \Delta)$ as a *curved V-data.*

Theorem 1 ([33, Thm. 1, Cor. 1])**.** *Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data. Then $\mathfrak{a}$ is a curved $L_\infty[1]$-algebra for the multibrackets $\{\emptyset\} := P\Delta$ and ($n \geq 1$)*

$$\{a_1,\dots,a_n\} = P[\dots[[\Delta,a_1],a_2],\dots,a_n]. \tag{3}$$

We obtain a $L_\infty[1]$-algebra exactly when $\Delta \in Ker(P)$.

When $\Delta \in Ker(P)$ there is actually a larger $L_\infty[1]$-algebra, which contains $\mathfrak{a}$ as in Thm. 1 as a $L_\infty[1]$-subalgebra.

Theorem 2 ([34, Thm. 2]). *Let $(L, \mathfrak{a}, P, \Delta)$ be a V-data, and denote $D := [\Delta, \cdot]\colon L \to L$. Then the space $L[1] \oplus \mathfrak{a}$ is a $L_\infty[1]$-algebra for the differential*

$$d(x[1], a) := (-(Dx)[1], P(x + Da)), \tag{4}$$

the binary bracket

$$\{x[1], y[1]\} = [x, y][1](-1)^{|x|} \in L[1],$$

and for $n \geq 1$:

$$\{x[1], a_1, \ldots, a_n\} = P[\ldots[x, a_1], \ldots, a_n] \in \mathfrak{a}, \tag{5}$$
$$\{a_1, \ldots, a_n\} = P[\ldots[Da_1, a_2], \ldots, a_n] \in \mathfrak{a}. \tag{6}$$

Here $x, y \in L$ and $a_1, \ldots, a_n \in \mathfrak{a}$. Up to permutation of the entries, all the remaining multibrackets vanish.

Notation 1.5. We will denote by

$$\mathfrak{a}^P_\Delta$$

and by

$$(L[1] \oplus \mathfrak{a})^P_\Delta$$

the $L_\infty[1]$-algebras produced by Thm. 1 and 2. We will also often consider the projection

$$P_\Phi := P \circ e^{[\cdot, \Phi]}\colon L \to \mathfrak{a}. \tag{7}$$

Remark 1.6. Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data and $\Phi \in \mathfrak{a}_0$ as above. Then Φ is a Maurer-Cartan element of $\mathfrak{a}^P_\Delta$ iff

$$P_\Phi \Delta = 0, \tag{8}$$

or equivalently $\Delta \in ker(P_\Phi)$. This follows immediately from eq. (3).

Remark 1.7. Let L' be a graded Lie subalgebra of L preserved by D (for example $L' = Ker(P)$). Then $L'[1] \oplus \mathfrak{a}$ is stable under the multibrackets of Thm. 2. We denote by $(L'[1] \oplus \mathfrak{a})^P_\Delta$ the induced $L_\infty[1]$-structure.

1.3 The main tool

The following statement is the main tool we develop. See [8, §1.3] for its proof. It is a statement about Maurer-Cartan elements of $L_\infty[1]$-algebras that arise as in Thm. 1. In the applications, these Maurer-Cartan elements will be the objects of interest, since they will correspond to morphisms, subalgebras, etc.

Theorem 3. *Let $(L, \mathfrak{a}, P, \Delta)$ be a filtered V-data and let $\Phi \in MC(\mathfrak{a}^P_\Delta)$. Then for all $\tilde{\Delta} \in L_1$ and $\tilde{\Phi} \in \mathfrak{a}_0$:*

$$\begin{cases} [\Delta + \tilde{\Delta}, \Delta + \tilde{\Delta}] = 0 \\ \Phi + \tilde{\Phi} \in MC(\mathfrak{a}^P_{\Delta + \tilde{\Delta}}) \end{cases} \Leftrightarrow (\tilde{\Delta}[1], \tilde{\Phi}) \in MC\Big((L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta\Big).$$

In this case, $\mathfrak{a}^P_{\Delta + \tilde{\Delta}}$ is a curved $L_\infty[1]$-algebra. It is a $L_\infty[1]$-algebra exactly when $\tilde{\Delta} \in Ker(P)$.

Remark 1.8. For any $\tilde{\Phi} \in \mathfrak{a}_0$ we have

$$\Phi + \tilde{\Phi} \text{ is a MC element of } \mathfrak{a}^P_\Delta \quad \Leftrightarrow \quad \tilde{\Phi} \text{ is a MC element of } \mathfrak{a}^{P_\Phi}_\Delta.$$

This is a well-known statement, saying that perturbations of a Maurer-Cartan element of $\mathfrak{a}^P_\Delta$ satisfy themselves a Maurer-Cartan equation, and is a particular case of the equivalence appearing in Thm. 3 (obtained setting $\tilde{\Delta} = 0$).

In the special case in which $\Delta = 0$ and $\Phi = 0$, we obtain the following corollary about the space of curved $L_\infty[1]$-algebra structures arising as in Thm. 1 and Maurer-Cartan elements in there:

Corollary 1.9. *Let $L, \mathfrak{a}, P$ such that $(L, \mathfrak{a}, P, 0)$ is a filtered V-data. The only non-vanishing multibrackets of $(L[1] \oplus \mathfrak{a})^P_0$, up to permutations of the entries, are*

$$\begin{aligned} d(x[1]) &= Px, \\ \{x[1], y[1]\} &= [x, y][1](-1)^{|x|}, \\ \{x[1], a_1, \dots, a_n\} &= P[\dots[x, a_1], \dots, a_n] \qquad \textit{for all } n \geq 1 \end{aligned}$$

where $x, y \in L$ and $a_1, \dots, a_n \in \mathfrak{a}$.

Its Maurer-Cartan elements are characterized by: for all $\tilde{\Delta} \in L_1$ and $\tilde{\Phi} \in \mathfrak{a}_0$

$$\begin{cases} [\tilde{\Delta}, \tilde{\Delta}] = 0 \\ \tilde{\Phi} \textit{ is a MC element of } \mathfrak{a}^P_{\tilde{\Delta}} \end{cases} \Leftrightarrow (\tilde{\Delta}[1], \tilde{\Phi}) \textit{ is a MC element of } (L[1] \oplus \mathfrak{a})^P_0.$$

1.4 Convergence issues

The left hand side of the Maurer-Cartan equation (2) is generally an infinite sum. In this subsection we review Getzler's notion of filtered $L_\infty[1]$-algebra [12], which guarantees that the above infinite sum converges. We show that simple assumptions on V-data ensure that the Maurer-Cartan equations of the (curved) $L_\infty[1]$-algebras we construct in Thm. 3 do converge.

Definition 1.10. Let V be a graded vector space. A *complete filtration* is a descending filtration by graded subspaces

$$V = \mathcal{F}^{-1}V \supset \mathcal{F}^0 V \supset \mathcal{F}^1 V \supset \dots$$

such that the canonical projection $V \to \underleftarrow{\lim} V/\mathcal{F}^n V$ is an isomorphism. Here

$$\underleftarrow{\lim} V/\mathcal{F}^n V := \{\vec{x} \in \Pi_{n \geq -1} V/\mathcal{F}^n V \ : \ P_{i,j}(x_j) = x_i \text{ when } i < j\},$$

where $P_{i,j} \colon V/\mathcal{F}^j V \longrightarrow V/\mathcal{F}^i V$ is the canonical projection induced by the inclusion $\mathcal{F}^j V \subset \mathcal{F}^i V$.

We define *Maurer-Cartan elements* to be $\Phi \in W_0 \cap \mathcal{F}^1 W$ for which the left hand side of eq. (2) vanishes.

Definition 1.11. Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data (Def. 1.4). We say that this curved V-data is *filtered* if there exists a complete filtration on L such that

a) The Lie bracket has filtration degree zero, i.e. $[\mathcal{F}^i L, \mathcal{F}^j L] \subset \mathcal{F}^{i+j} L$ for all $i, j \geq -1$,

b) $\mathfrak{a}_0 \subset \mathcal{F}^1 L$,

c) the projection P has filtration degree zero, i.e. $P(\mathcal{F}^i L) \subset \mathcal{F}^i L$ for all $i \geq -1$.

See [8, §1.3] for the proof of the following proposition.

Proposition 1.12. *Let $(L, \mathfrak{a}, P, \Delta)$ be a filtered, curved V-data. Then for every $\Phi \in MC(\mathfrak{a}^P_\Delta) \subset \mathfrak{a}_0$:*

1) the projection $P_\Phi := P \circ e^{[\cdot,\Phi]} \colon L \to \mathfrak{a}$ is well-defined and has filtration degree zero.

2) the curved $L_\infty[1]$-algebra $\mathfrak{a}^{P_\Phi}_\Delta$ given by Thm. 1 is filtered by $\mathcal{F}^n\mathfrak{a} := \mathcal{F}^n L \cap \mathfrak{a}$. Further, the sum on the left hand side of eq. (2) converges for any degree zero element a of $\mathfrak{a}$.

3) if $\Delta \in ker(P)$: the $L_\infty[1]$-algebra $(L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$ given by Thm. 2 is filtered by $\mathcal{F}^n(L[1] \oplus \mathfrak{a}) := (\mathcal{F}^n L)[1] \oplus \mathcal{F}^n\mathfrak{a}$. Further, the sum on the left hand side of eq. (2) converges for any degree zero element $(x[1], a)$ of $L[1] \oplus \mathfrak{a}$.

A common way to deal with convergence issues is to work formally (i.e. in terms of power series in a formal variable ε). The following is the analogue of Prop. 1.12 in the formal setting:

Lemma 1.13. *Let $(L, \mathfrak{a}, P, \Delta)$ be a curved V-data (not necessarily filtered). Let $\Phi \in \mathfrak{a}_0 \otimes \varepsilon \cdot \mathbb{R}[[\varepsilon]]$.*

1) for the Maurer-Cartan equation of the curved $L_\infty[1]$-algebra $(\mathfrak{a} \otimes \mathbb{R}[[\varepsilon]])^{P_\Phi}_\Delta$ the following holds: the sum on the left hand side of eq. (2) converges for any element a of $\mathfrak{a}_0 \otimes \varepsilon \cdot \mathbb{R}[[\varepsilon]]$.

2) if $\Delta \in ker(P)$, for the Maurer-Cartan equation of the $L_\infty[1]$-algebra $\big((L[1] \oplus \mathfrak{a}) \otimes \mathbb{R}[[\varepsilon]]\big)^{P_\Phi}_\Delta$ the following holds: the sum on the left hand side of eq. (2) converges for any element $(x[1], a) \in (L[1] \oplus \mathfrak{a})_0 \otimes \varepsilon \cdot \mathbb{R}[[\varepsilon]]$.

Proof. One checks easily that the following is a curved V-data:

- the graded Lie algebra $L \otimes \mathbb{R}[[\varepsilon]]$
- its abelian subalgebra $\mathfrak{a} \otimes \mathbb{R}[[\varepsilon]]$
- the natural projection $P \colon L \otimes \mathbb{R}[[\varepsilon]] \to \mathfrak{a} \otimes \mathbb{R}[[\varepsilon]]$
- Δ,

where the the first three structures are defined by $\mathbb{R}[[\varepsilon]]$-linear extension. The natural complete filtration $\{\mathcal{F}^n\}_{n \geq 0}$ on the vector space $L \otimes \mathbb{R}[[\varepsilon]]$ by $\mathcal{F}^n := L \otimes \varepsilon^n \mathbb{R}[[\varepsilon]]$ satisfies conditions a), c) of Def. 1.11. It does not satisfy condition b), however the proof of Prop. 1.12, applied to the above curved V-data, goes through whenever Φ and a lie in $\mathfrak{a}_0 \otimes \varepsilon \cdot \mathbb{R}[[\varepsilon]]$. □

Remark 1.14. Notice that the curved $L_\infty[1]$-algebra $(\mathfrak{a} \otimes \mathbb{R}[[\varepsilon]])^P_\Delta$ is canonically isomorphic to $(\mathfrak{a}^P_\Delta) \otimes \mathbb{R}[[\varepsilon]]$.

1.5 Equivalences of Maurer-Cartan elements

Let W be an $L_\infty[1]$-algebra. On $MC(W)$, the set of Maurer-Cartan elements, there is a canonical involutive (singular) distribution $\mathcal{D}$ which induces an equivalence relation on $MC(W)$ known as *gauge equivalence*. More precisely, each $z \in W_{-1}$ defines a vector field $\mathcal{Y}^z$ on W_0, whose value at $m \in W_0$ is[1]

$$\mathcal{Y}^z|_m := dz + \{z, m\} + \frac{1}{2!}\{z, m, m\} + \frac{1}{3!}\{z, m, m, m\} + \dots . \tag{9}$$

This vector field is tangent to $MC(W)$. The distribution at the point $m \in MC(W)$ is defined as $\mathcal{D}|_m = \{\mathcal{Y}^z|_m : z \in W_{-1}\}$.

Remark 1.15. When the differential d vanishes, the Jacobiator of the binary bracket $\{\cdot,\cdot\}$ is zero. Hence $\{\cdot,\cdot\}$ makes the vector space W_{-1} into an ordinary Lie algebra, and the assignment $W_{-1} \to \chi_0(W_0), z \mapsto (\mathcal{Y}^z)_{lin} := \{z, \cdot\} \in \chi_0(W_0)$ to the linear part of $\mathcal{Y}^z$ is a Lie algebra morphism.

Consider in particular the $L_\infty[1]$-subalgebra $ker(P)[1] \oplus \mathfrak{a}$ of the $L_\infty[1]$-algebra of Cor. 1.9. Notice that the differential vanishes, so Remark 1.15 applies. The vector field associated to a degree -1 element $z = (z_L[1], z_\mathfrak{a}) \in ker(P)[1] \oplus \mathfrak{a}$, evaluated at $m = (m_L[1], m_\mathfrak{a}) \in MC(ker(P)[1] \oplus \mathfrak{a})$, reads

$$\mathcal{Y}^z|_m = [z_L, m_L][1] + \sum_{n\geq 1} \frac{1}{n!} P[[z_L, \underbrace{m_\mathfrak{a}], \dots, m_\mathfrak{a}]}_{n \text{ times}} + \sum_{n \geq 1} \frac{1}{(n-1)!} P[[[m_L, z_\mathfrak{a}], \underbrace{m_\mathfrak{a}], \dots, m_\mathfrak{a}]}_{n-1 \text{ times}} \tag{10}$$

where the square bracket is the graded Lie algebra structure on L.

We will display explicitly the equivalence relation induced on morphisms between Lie algebras in §2.1. It turns out that the equivalence classes coincide with the orbits of a group action.

2 Applications to Lie theory

In this section we apply the machinery developed in the previous section to instances in Lie theory. For the examples we threat here, procedures to recover the $L_\infty[1]$-algebras governing simultaneous deformations are known [23][7], but often are not exhibited in explicit form in the literature. Using our machinery, we make the $L_\infty[1]$-algebras structures quite explicit. The results of §2.1 recover a theorem in [7]. We mention further that the results we obtained §2.2 have been recently extended by Ji from the setting of Lie algebras to that of Lie algebroids [15].

We refer the reader to Appendix A.1 for the background material needed in §2.1- 2.3., and to Appendix A.2 for that needed in §2.4 - 2.5.

[1]The infinite sum (9) is guaranteed to converge if W is filtered and $W_{-1} \subset \mathcal{F}^1 W$, see §1.4. In the example we consider in this paper, this sum is actually finite, see eq. (20).

2.1 Lie algebra morphisms.

Let $(U, [\cdot,\cdot]_U)$ and $(V, [\cdot,\cdot]_V)$ be finite dimensional Lie algebras. We show that the deformations of Lie algebra morphisms $U \to V$ are ruled by a DGLA, recovering classical results of Nijenhuis and Richardson [25], and that more generally the simultaneous deformations of the Lie algebra structures and Lie algebra morphisms are ruled by a L_∞-algebra, recovering a theorem in [7] by the first author, Markl and Yau. The set-up of this subsection is a special case of the one of §2.5. We consider the simple instance of Lie algebras separately for the sake of concreteness and clarity of exposition. Further, we discuss equivalences.

We consider the graded manifold $(U \times V)[1]$, and encode the above data as vector fields on this graded manifold. See Appendix A.1 for some basic notions on graded manifolds and the notation; in particular $\chi(U[1])$ denotes the space of vector fields on $U[1]$, and $\iota\colon U \to \chi_{-1}(U[1])$ identifies elements of U with constant vector fields. We adopt the following conventions:

- The Lie bracket $[\cdot,\cdot]_U$ is encoded by the vector field $Q_U \in \chi_1(U[1])$ defined by $[[Q_U, \iota_X], \iota_Y] = \iota_{[X,Y]_U}$ for all $X, Y \in U$. The Jacobi identity for $[\cdot,\cdot]_U$ is equivalent to this vector field being homological (i.e., $[Q_U, Q_U] = 0$)
- A linear map $\phi\colon U \to V$ is encoded by $\Phi \in \chi_0((U \times V)[1])$ defined by $[\Phi, \iota_X] = \iota_{\phi(X)}$ for all $X \in U$.

Remark 2.1. We give coordinate expressions for the vector fields Q_U, Q_V, Φ. Choose a basis of U, giving rise to coordinates $\{u_i\}$ on $U[1]$, and similarly choosing a basis of V get coordinates $\{v_\alpha\}$ on $V[1]$. Then

$$Q_U = -\frac{1}{2} c_{ij}^k u_i u_j \frac{\partial}{\partial u_k}, \qquad Q_V = -\frac{1}{2} d_{\alpha\beta}^\gamma v_\alpha v_\beta \frac{\partial}{\partial v_\gamma}, \qquad \Phi = -A_{l\eta} u_l \frac{\partial}{\partial v_\eta} \tag{11}$$

where c_{ij}^k and $d_{\alpha\beta}^\gamma$ are the structural constants of the Lie algebras U and V respectively and $A_{l\eta}$ is the matrix representing ϕ in the chosen basis.

The map $\phi\colon U \to V$ is a Lie algebra morphism exactly when

$$[Q_U, \Phi] + \frac{1}{2}[[Q_V, \Phi], \Phi] = 0, \tag{12}$$

see for example [25, p. 176].

Lemma 2.2. *The following quadruple forms a V-data:*

- *the graded Lie algebra* $L := \chi((U \times V)[1])$
- *its abelian subalgebra* $\mathfrak{a} := C(U[1]) \otimes V[1]$
- *the natural projection* $P\colon L \to \mathfrak{a}$ *with kernel*

$$ker(P) = \Big(C(U[1]) \otimes C_{\geq 1}(V[1]) \otimes V[1]\Big) \;\oplus\; \Big(C(U[1] \times V[1]) \otimes U[1]\Big)$$

- $\Delta := Q_U + Q_V$,

hence by Thm. 1 we obtain a $L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$. For every linear map $\phi\colon U \to V$ we have: $\Phi \in \mathfrak{a}_0$ is a Maurer-Cartan element in $\mathfrak{a}^P_\Delta$ iff ϕ is a Lie algebra morphism.

Proof. $Ker(P)$ is a Lie subalgebra of L. This can be seen in coordinates, or noticing that the kernel consists exactly of vector fields on $(U \times V)[1]$ which are tangent to $(U \times \{0\})[1]$. Further we have $[\Delta, \Delta] = [Q_U, Q_U] + [Q_V, Q_V] = 0$. Hence the above quadruple forms a V-data.

The $L_\infty[1]$-structure induced on $\mathfrak{a}$ by Thm. 1 is given by the multibrackets $P[[[Q_U + Q_V, \cdot], \cdots], \cdot]$. One computes easily in coordinates using (11) that $P[Q_V, \cdot]$, $[[Q_U, \cdot], \cdot]$ and $[[[Q_V, \cdot], \cdot], \cdot]$ vanish when applied to elements of $\mathfrak{a}$. Hence only the unary and binary brackets are non-zero, and they are given by

$$[Q_U, \cdot]$$
$$[[Q_V, \cdot], \cdot]$$

respectively. Therefore the Maurer-Cartan equation of $\mathfrak{a}^P_\Delta$ is given by (12). $\square$

Lemma 2.2 allows us to apply Thm. 3 (and Rem. 1.8). Hence we deduce:

Corollary 2.3. *Let U, V finite dimensional Lie algebras and $\phi\colon U \to V$ a morphism. Let $(L, \mathfrak{a}, P, \Delta)$ as in Lemma 2.2.*

1) Let $\tilde{\phi}\colon U \to V$ be a linear map. Then

$$\phi + \tilde{\phi} \text{ is a Lie algebra morphism} \quad \Leftrightarrow \quad \tilde{\Phi} \text{ is a MC element of } \mathfrak{a}^{P_\Phi}_\Delta.$$

2) For all quadratic vector fields $\tilde{Q}_U$ on $U[1]$ and $\tilde{Q}_V$ on $V[1]$ and for all linear maps $\tilde{\phi}\colon U \to V$:

$$\begin{cases} Q_U + \tilde{Q}_U \text{ and } Q_V + \tilde{Q}_V \text{ define Lie algebra structures on } U \text{ and } V \\ \phi + \tilde{\phi} \text{ is a Lie algebra morphism between these new Lie algebra structures} \end{cases}$$
$$\Leftrightarrow ((\tilde{Q}_U + \tilde{Q}_V)[1], \tilde{\Phi}) \text{ is a MC element of } (L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta.$$

Remark 2.4. We check that $(L, \mathfrak{a}, P, \Delta)$ is filtered V-data (Def. 1.11), as this is a hypothesis in Thm. 3. We have a direct sum decomposition $L = \oplus_{k \geq -1} L^k$ where $L^k := C_{k+1}(U[1]) \otimes C(V[1]) \otimes U[1] \oplus C_k(U[1]) \otimes C(V[1]) \otimes V[1]$. In other words, L^k is spanned by monomials in L whose total number of u's and $\frac{\partial}{\partial v}$'s, in coordinates, is exactly $k+1$. Then $\mathcal{F}^n L := \oplus_{k \geq n} L^k$ is a complete filtration of the vector space L. One checks easily that $(L, \mathfrak{a}, P, \Delta)$ is filtered V-data.

An alternative way to check that there are no convergence issues for $e^{[\cdot, \Phi]}$ and the Maurer-Cartan equations appearing in Cor. 2.3 is to recall that $U \times V$ is finite dimensional and use a variant of Lemma 2.6 below.

2.1.1 Explicit expressions for the multibrackets

In this subsection we make more explicit the structures of $\mathfrak{a}^{P_\Phi}_\Delta$ and $(L'[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$, where $L' \subset L$ is specified just after Lemma 2.5.

Given a morphism of Lie algebras $\phi\colon U \to V$, the associated *Richardson-Nijenhius DGLA* is given by $\oplus_i \wedge^i U^* \otimes V$, the differential being the Chevalley-Eilenberg differential of U with values in the module V (the module structure is given by $e \in U \mapsto [\phi(e), \cdot]_V$) and the bracket being the Lie bracket on V combined with the wedge product on $\wedge U^*$ (see [25, p. 175-6] or [6, §2.3]).

Lemma 2.5. *$\mathfrak{a}_\Delta^{P_\Phi}$ is the suspension of the Richardson-Nijenhius DGLA.*

Proof. The n-ary bracket of $\mathfrak{a}_\Delta^{P_\Phi}$, evaluated on $a_1, \dots, a_n \in \mathfrak{a}$ is

$$P_\Phi[[[Q_U + Q_V, a_1], \cdots], a_n]$$

One computes easily in coordinates that only unary and binary brackets are non-zero, and they are given by

$$P[Q_U + [Q_V, \Phi], \cdot] = [Q_U + [Q_V, \Phi], \cdot] \tag{13}$$

$$P[[Q_V, \cdot], \cdot] = [[Q_V, \cdot], \cdot]. \tag{14}$$

respectively. The r.h.s. of (13) is exactly the Chevalley-Eilenberg differential of the Lie algebra U with values in the module V. The r.h.s. of (14) is given by the Lie bracket on V combined with the wedge product on $\wedge U^*$. Hence we obtain the suspension of the Nijenhuis-Richardson DGLA. □

Up to this point we only looked at deformations of the morphism $\phi\colon U \to V$. Now we also deform the Lie algebra structures on the vector spaces U and V.

Define $L' := \chi(U[1]) \oplus \chi(V[1]) \subset L$. By Thm. 3 and Rem. 1.7 we obtain an $L_\infty[1]$-algebra $(L'[1] \oplus \mathfrak{a})_\Delta^{P_\Phi}$, governing the simultaneous deformations of the Lie algebra structures on U, V and of the morphisms.

Lemma 2.6. *$(L'[1] \oplus \mathfrak{a})_\Delta^{P_\Phi}$ has multibrackets of order up to $dim(V) + 1$. Its Maurer-Cartan equation is cubic, given by eq.* (17)*,* (18) *and* (19) *below.*

Proof. We write down explicitly the multibrackets of $(L'[1] \oplus \mathfrak{a})_\Delta^{P_\Phi}$, as given in Thm. 2. We denote by $\tilde{Q}_U^i, \tilde{Q}_V^i$ and $\tilde{\Phi}^i$ general (homogeneous) elements of $\chi(U[1]), \chi(V[1])$ and $\mathfrak{a}$ respectively ($i = 1, 2, \dots$). The multibrackets involving only $\tilde{\Phi}$ are given exactly by (13) and (14) since $\mathfrak{a}_\Delta^{P_\Phi}$ is a L_∞-subalgebra of $(L'[1] \oplus \mathfrak{a})_\Delta^{P_\Phi}$. Explicitly, they are

$$d(\tilde{\Phi}) = [Q_U + [Q_V, \Phi], \tilde{\Phi}] \in \mathfrak{a}$$

and

$$\{\tilde{\Phi}^1, \tilde{\Phi}^2\} = [[Q_V, \tilde{\Phi}^1], \tilde{\Phi}^2] \in \mathfrak{a}.$$

Now we compute the multibrackets involving at least one of $\tilde{Q}_U[1]$ or $\tilde{Q}_V[1]$. For the differential we have in $L[1] \oplus \mathfrak{a}$:

$$d(\tilde{Q}_U[1]) = -[Q_U + Q_V, \tilde{Q}_U][1] + P_\Phi(\tilde{Q}_U) = -[Q_U, \tilde{Q}_U][1] + [\tilde{Q}_U, \Phi]$$

$$d(\tilde{Q}_V[1]) = -[Q_U + Q_V, \tilde{Q}_V][1] + P_\Phi(\tilde{Q}_V) = -[Q_V, \tilde{Q}_V][1] + \frac{1}{k!}[[\dots[\tilde{Q}_V, \underbrace{\Phi], \dots], \Phi]}_{k}$$

where $k = |\tilde{Q}_V| + 1$. For the binary bracket we have

$$\begin{aligned} \{(\tilde{Q}_U^1 + \tilde{Q}_V^1)[1], (\tilde{Q}_U^2 + \tilde{Q}_V^2)[1]\} &= (-1)^{|\tilde{Q}_U^1 + \tilde{Q}_V^1|}\Big([\tilde{Q}_U^1, \tilde{Q}_U^2] + [\tilde{Q}_V^1, \tilde{Q}_V^2]\Big)[1] \in L[1] \\ \{\tilde{Q}_U[1], \tilde{\Phi}\} &= P_\Phi[\tilde{Q}_U, \tilde{\Phi}] = [\tilde{Q}_U, \tilde{\Phi}] \in \mathfrak{a} \\ \{\tilde{Q}_V[1], \tilde{\Phi}\} &= P_\Phi[\tilde{Q}_V, \tilde{\Phi}] \in \mathfrak{a}. \end{aligned} \tag{15}$$

From (15) it follows that the only non-zero n-brackets with $n \geq 3$ are

$$\{\tilde{Q}_V[1], \tilde{\Phi}^1, \dots, \tilde{\Phi}^n\} = P_\Phi[[\tilde{Q}_V, \tilde{\Phi}^1], \dots, \tilde{\Phi}^n] \in \mathfrak{a}. \tag{16}$$

In coordinates it is clear that the operation $[\cdot, \tilde{\Phi}]$ sends $C(U[1]) \otimes C_i(V[1]) \otimes V[1]$ to $C(U[1]) \otimes C_{i-1}(V[1]) \otimes V[1]$. As $\tilde{Q}_V \in \chi(V[1]) \cong \sum_{i=1}^{dim(V)} C_i(V[1]) \otimes V[1]$, it is clear from eq. (16) that all n-brackets vanish for $n > dim(V) + 1$.

To write down the Maurer-Cartan elements , we can use eq. (2) and the formulae for the multibrackets derived above. Alternatively, by virtue of Cor. 2.3, we know that Maurer-Cartan elements $\tilde{Q} = \tilde{Q}_U[1] + \tilde{Q}_V[1] + \tilde{\Phi}$ are characterized by the equations $[Q_U + \tilde{Q}_U, Q_U + \tilde{Q}_U] = 0$, $[Q_V + \tilde{Q}_V, Q_V + \tilde{Q}_V] = 0$ and by the equation obtained replacing Q_U by $Q_U + \tilde{Q}_U$ (and similarly for Q_V,Φ) in eq. (12) . The first two equations are equivalent to

$$[Q_U, \tilde{Q}_U] + \frac{1}{2}[\tilde{Q}_U, \tilde{Q}_U] = 0 \tag{17}$$

$$[Q_V, \tilde{Q}_V] + \frac{1}{2}[\tilde{Q}_V, \tilde{Q}_V] = 0 \tag{18}$$

while the third equation reads

$$\begin{aligned} 0 =& [\tilde{Q}_U, \Phi] + \frac{1}{2}[[\tilde{Q}_V, \Phi], \Phi] + [Q_U + [Q_V, \Phi], \tilde{\Phi}] \\ &+ [\tilde{Q}_U, \tilde{\Phi}] + [[\tilde{Q}_V, \tilde{\Phi}], \Phi] + \frac{1}{2}[[Q_V, \tilde{\Phi}], \tilde{\Phi}] \\ &+ \frac{1}{2}[[\tilde{Q}_V, \tilde{\Phi}], \tilde{\Phi}]. \end{aligned} \tag{19}$$

□

2.1.2 Equivalences of Lie algebras morphisms

Consider the $L_\infty[1]$-algebra whose Maurer-Cartan elements are pairs of Lie algebra structures and morphisms between them, that is, the $L_\infty[1]$-algebra $\mathcal{L} := (L'[1] \oplus \mathfrak{a})^P_{\Delta=0}$ as in Cor. 1.9. Here we discuss the natural equivalence on the set of Maurer-Cartan elements, see §1.5.

Elements of $\mathcal{L}_{-1}$ are of the form

$$z = (z_U[1], z_V[1], z_\mathfrak{a}) \in \ \chi_0(U[1])[1] \ \oplus \ \chi_0(V[1])[1] \ \oplus \ V[1].$$

Restricting the binary bracket $\{\cdot, \cdot\}_2$ to $\mathcal{L}_{-1}$ and using the identifications at the beginning of §2.1 we obtain the ordinary Lie algebra

$$End(U) \times (End(V) \ltimes V)$$

where $End(U)$ and $End(V)$ are endowed with the commutator bracket, V is abelian and $[A, f] = Af \in V$ for $A \in End(V)$ and $f \in V$.

Maurer-Cartan elements lie in $\mathcal{L}_0$, so they are of the form

$$m = (m_U[1], m_V[1], m_{\mathfrak{a}}) \in \ \chi_1(U[1])[1] \ \oplus \ \chi_1(V[1])[1] \ \oplus \ (U[1])^* \otimes V[1],$$

and as described at the beginning of §2.1 their components correspond respectively to a Lie bracket $[\cdot,\cdot]_{m_U}$ on U, a Lie bracket $[\cdot,\cdot]_{m_V}$ on V, and a Lie algebra morphism $\phi\colon U \to V$. By degree reasons eq. (10) reads simply

$$\begin{aligned} \mathcal{Y}^z|_m = \quad & [z_U, m_U][1] \ \oplus \ [z_V, m_V][1] \ \oplus \ [z_U + z_V, m_{\mathfrak{a}}] + [[m_V, z_{\mathfrak{a}}], m_{\mathfrak{a}}] \\ \in & T_z\Big(\chi_1(U[1])[1] \ \oplus \ \chi_1(V[1])[1] \ \oplus \ (U[1])^* \otimes V[1]\Big). \end{aligned} \tag{20}$$

The assignment $z \mapsto \mathcal{Y}^z$ vector field is not a Lie algebra action: $z^1 = (0, 0, z^1_{\mathfrak{a}})$ and $z^2 = (0, 0, z^2_{\mathfrak{a}})$ commute, but the vector fields $\mathcal{Y}^{z^1}$ and $\mathcal{Y}^{z^2}$ do not commute. However restricting suitably we obtain an infinitesimal action, which integrates to the group action of symmetries given in [6, §3]:

Proposition 2.7. *The assignment $End(U) \times End(V) \to \chi(MC(\mathcal{L})), z \mapsto \mathcal{Y}^z$ is a Lie algebra morphism. It integrates to the group action*

$$\begin{aligned} &\Big(GL(U) \times GL(V)\Big) \times MC(\mathcal{L}) \to MC(\mathcal{L}) \\ &\quad (g, h)\ ,\ \Big([\cdot,\cdot]_{m_U}, [\cdot,\cdot]_{m_U}, \phi\Big) \mapsto \Big(g^*([\cdot,\cdot]_{m_U}), h^*([\cdot,\cdot]_{m_V}), h \circ \phi \circ g^{-1}\Big). \end{aligned}$$

Here the Lie bracket $g^*([\cdot,\cdot]_{m_U})$ is defined as $g[g^{-1}\cdot, g^{-1}\cdot]_{m_U}$, and similarly for $h^*([\cdot,\cdot]_{m_V})$.

The equivalence classes induced by the singular distribution $\mathcal{D} := \{\mathcal{Y}^z : z \in \mathcal{L}_{-1}\}$ on MC agree with the orbits of the this action.

Proof. Notice that for $z \in End(U) \times End(V)$ the vector field $\mathcal{Y}^z$ is linear, hence $z \mapsto \mathcal{Y}^z$ is a Lie algebra morphism by Remark 1.15. We compute the integral curve of $\mathcal{Y}^z$ starting at $m = (m_U[1], m_V[1], m_{\mathfrak{a}}) \in MC(\mathcal{L})$.

The first component of $\mathcal{Y}^z$ is $[z_U, \cdot][1]$. Its integral curve starting at $m_U[1]$ is $t \mapsto e^{t[z_U,\cdot]}m_U[1]$, since the latter forms a 1-parameter group and differentiates to $[z_V, \cdot]$ at time zero. The Lie bracket on U induced by $e^{[z_U,\cdot]}m_U[1]$ is $(exp(z_U))^*([\cdot,\cdot]_{m_U})$ where $exp(z_U)$ is the usual matrix exponential of $z_U \in \mathfrak{gl}(U)$ (this follows from the fact that $e^{[z_U,\cdot]}$ is an automorphism of $[\cdot,\cdot]$). The same argument applies to the second component of $\mathcal{Y}^z$.

For the third component, the integral curve of $[z_U + z_V, \cdot]$ starting at $m_{\mathfrak{a}}$ is $t \mapsto e^{t[z_U+z_V,\cdot]}m_{\mathfrak{a}}$. The element $e^{[z_U+z_V,\cdot]}m_{\mathfrak{a}} \in (U[1])^* \otimes V[1]$ corresponds to $exp(z_V) \circ \phi \circ exp(-z_U)\colon U \to V$. This shows that the group action in the statement of this proposition integrates the given Lie algebra action.

For the last statement we fix $m \in MC(\mathcal{L})$ and show that

$$\mathcal{D}_m = \{\mathcal{Y}^z|_m : z = (z_U[1], z_V[1], 0)\}.$$

To this aim, just notice that $\mathcal{Y}^{(0,0,z_{\mathfrak{a}})}|_m = \mathcal{Y}^{(0,[m_V,z_{\mathfrak{a}}],0)}|_m$ for all $z_{\mathfrak{a}} \in V[1]$, as a consequence of $[m_V, m_V] = 0$. ◻

2.2 Subalgebras of Lie algebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $U \subset \mathfrak{g}$ a Lie subalgebra. We study deformations of the Lie algebra structure on $\mathfrak{g}$ and of the subspace U as a Lie subalgebra, similarly to Richardson [27].

At first, let $U \subset \mathfrak{g}$ be simply a subspace. We denote by $Q_{\mathfrak{g}} \in \chi(\mathfrak{g}[1])$ the homological vector field encoding the Lie algebra structure on $\mathfrak{g}$. Choose a subspace V in $\mathfrak{g}$ complementary to U. Given a linear map $\phi\colon U \to V$, we view it as an element $\Phi \in C_1(U[1]) \otimes \chi_{-1}(V[1]) \subset \chi_0(\mathfrak{g}[1])$ defined by $[\Phi, \iota_X] = \iota_{\phi(X)}$ for all $X \in U$.

Lemma 2.8. *The following quadruple forms a curved V-data:*

- *the graded Lie algebra* $L := \chi(\mathfrak{g}[1])$
- *its abelian subalgebra* $\mathfrak{a} := C(U[1]) \otimes V[1]$
- *the natural projection* $P\colon L \to \mathfrak{a}$ *with kernel*

$$ker(P) = \Big(C(U[1]) \otimes C_{\geq 1}(V[1]) \otimes V[1]\Big) \;\oplus\; \Big(C(\mathfrak{g}[1]) \otimes U[1]\Big)$$

- $\Delta := Q_{\mathfrak{g}}$,

hence by Thm. 1 we obtain a curved $L_\infty[1]$*-structure* $\mathfrak{a}^P_\Delta$.

$\Phi \in \mathfrak{a}_0$ *is a MC element in* $\mathfrak{a}^P_\Delta$ *iff* $graph(\phi)$ *is a Lie subalgebra of* $\mathfrak{g}$.

Further, the above quadruple forms a V-data iff U *is a Lie subalgebra of* $\mathfrak{g}$.

Proof. To show that the above quadruple forms a curved V-data proceed as in the proof of Lemma 2.2.

Rem. 1.6 says that Φ is a Maurer-Cartan element in $\mathfrak{a}^P_\Delta$ iff $e^{-[\Phi,\cdot]}Q_{\mathfrak{g}} \in ker(P)$. This condition is equivalent to asking that for all $X, Y \in U$:

$$\Big[\Big[e^{-[\Phi,\cdot]}Q_{\mathfrak{g}}, \iota_X\Big], \iota_Y\Big] \in U[1]$$

Using the fact that $e^{-[\Phi,\cdot]}$ is a Lie algebra automorphism of L (to pull it out of the brackets) and that $e^{[\Phi,\cdot]}\iota_X = \iota_X + [\Phi, \iota_X] = \iota_{X+\phi(X)}$, we see that the above is equivalent to

$$[X + \phi(X), Y + \phi(Y)] \in \{Z + \phi(Z) : Z \in U\} = graph(\phi),$$

i.e. to $graph(\phi)$ being a Lie subalgebra of $\mathfrak{g}$.

The last statement can be proven as follows: $Q_{\mathfrak{g}} \in ker(P)$ is equivalent to $[[Q_{\mathfrak{g}}, \iota_X], \iota_Y] \in U[1]$ for all $X, Y \in U$, which in turn means that U is a Lie subalgebra of $\mathfrak{g}$. (Alternatively, it follows from the above noticing that 0 is a Maurer-Cartan element of $\mathfrak{a}^P_\Delta$ iff $PQ_{\mathfrak{g}} = 0$.) ☐

Lemma 2.8 allow us to apply Thm. 3 with $\Phi = 0$. We deduce:

Corollary 2.9. *Let* $\mathfrak{g}$ *be a Lie algebra,* $U \subset \mathfrak{g}$ *a Lie subalgebra. Choose a subspace* $V \subset \mathfrak{g}$ *complementary to* U*, and let* $(L, \mathfrak{a}, P, \Delta)$ *be the V-data as in Lemma 2.8.*

For all $\tilde{Q}_{\mathfrak{g}} \in L_1$ *and for all linear maps* $\tilde{\phi}\colon U \to V$*:*

$$\begin{cases} Q_{\mathfrak{g}} + \tilde{Q}_{\mathfrak{g}} \textit{ defines a Lie algebra structure on } \mathfrak{g} \\ graph(\tilde{\phi}) \textit{ is a Lie subalgebra of it} \end{cases}$$
$$\Leftrightarrow (\tilde{Q}_{\mathfrak{g}}[1], \tilde{\Phi}) \textit{ is a MC element of } (L[1] \oplus \mathfrak{a})^P_\Delta.$$

Remark 2.10. The proof that $(L, \mathfrak{a}, P, \Delta)$ is a filtered V-data is given in Remark 2.4.

Remark 2.11. By Cor. 2.9, the Maurer-Cartan elements of $(L[1] \oplus \mathfrak{a})^P_\Delta$ are in bijection with deformations of the Lie algebra structure on $\mathfrak{g}$ and deformations of the subspace U as a Lie subalgebra.

Applying Cor. 2.3 to the Lie algebra U, to the Lie algebra $\mathfrak{g}$ and to the inclusion $i \colon U \hookrightarrow \mathfrak{g}$, we obtain an $L_\infty[1]$-algebra whose Maurer-Cartan elements are deformations of the Lie algebra structure on $\mathfrak{g}$ and deformations of i to linear maps $i + \tilde{i} \colon U \to \mathfrak{g}$ whose image is a Lie subalgebra of the new Lie algebra structure on $\mathfrak{g}$. Notice that the two Maurer-Cartan sets are quite different, as different maps $i + \tilde{i}$ can have the same image.

2.3 Lie bialgebra morphisms.

Let U and V be Lie bialgebras. We show that the simultaneous deformations of the Lie bialgebra structures and Lie bialgebra morphisms are ruled by some L_∞-algebra.

Definition 2.12. A finite dimensional vector space U is a *Lie bialgebra* if U is endowed with a Lie algebra structure, the dual U^* is endowed with a Lie algebra structure $[\cdot,\cdot]_{U^*}$, and the Chevalley-Eilenberg differential of U is a graded derivation of $[\cdot,\cdot]_{U^*}$ (or more precisely, of its extension to $\wedge U^*$).

A *morphism* between from a Lie bialgebra U to a Lie bialgebra V is a Lie algebra morphism $\phi \colon U \to V$ such that its dual $\phi^* \colon V^* \to U^*$ is also a Lie algebra morphism (see for instance [2]).

In order to rephrase the above definitions, we recall few notions from graded geometry. Let U be a vector space. The graded manifold $\mathcal{M} := T^*[2]U[1] = U[1] \times U^*[1]$ is symplectic, hence the space of functions is endowed with a degree -2 Poisson bracket[2]. Explicitly, the degree k functions are[3] $C_k(\mathcal{M}) = \wedge^k(U^* \times U)$. If we choose a basis for U, giving rise to degree 1 coordinates u_i on $U[1]$ and degree 1 coordinates on $U^*[1]$ which we denote by $\frac{\partial}{\partial u_i}$, the Poisson bracket is given by

$$\{u_i, u_j\} = 0, \quad \{\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\} = 0, \quad \{u_i, \frac{\partial}{\partial u_j}\} = \delta_{ij} = \{\frac{\partial}{\partial u_j}, u_i\}.$$

Notice that $C(\mathcal{M})$ is not only graded but actually bigraded, by $C_{(i,j)}(\mathcal{M}) = \wedge^i U^* \otimes \wedge^j U$.

Since the Poisson bracket on $C(\mathcal{M})$ has degree -2, it follows that $C(\mathcal{M})[2]$ is a graded Lie algebra. There is a canonical (degree preserving) embedding

$$\chi(U[1]) \hookrightarrow C(\mathcal{M})[2],$$

whose image consists exactly of $\oplus_i C_{(i,1)}(\mathcal{M})$ (the fiber-wise linear functions on $\mathcal{M} = T^*[2]U[1]$). The embedding also preserves the brackets by [28, Lemma 3.3.1], i.e., it is an embedding of graded Lie algebras. Notice that there is a canonical symplectomorphism $T^*[2]U[1] = U[1] \times U^*[1] \cong U^*[1] \times U[1] = T^*[2]U^*[1]$, which provides a canonical embedding of graded Lie algebras $\chi(U^*[1]) \hookrightarrow C(\mathcal{M})[2]$.

[2] This bracket is sometimes called "big bracket".

[3] Here we use $\wedge$ to denote the ordinary exterior power, and regard elements of U and U^* as having degree one.

We can now state, following [28, §3.1]: a Lie bialgebra structure on U is equivalent to an element $Q_U \in C_{(2,1)}(\mathcal{M})$ and an element $Q_{U^*} \in C_{(1,2)}(\mathcal{M})$ such that $Q_U + Q_{U^*}$ commutes with itself w.r.t. $\{\cdot,\cdot\}$, or equivalently so that $Q_U - Q_{U^*}$ self-commutes.

Further, if U and V are Lie bialgebras and given a linear map $\phi\colon U \to V$, consider the corresponding element $\Phi \in \chi((U \times V)[1]) \subset C(T^*[2](U \times V)[1])[2]$ as at the beginning of this section. Notice that the element of $C(T^*[2](U\times V)[1])[2]$ associated to ϕ^* is $-\Phi$. Using eq. (12) we see that ϕ is a morphism of Lie bialgebras iff

$$\{Q_U, \Phi\} + \frac{1}{2}\{\{Q_V, \Phi\}, \Phi\} = 0, \tag{21}$$

$$\{Q_{V^*}, -\Phi\} + \frac{1}{2}\{\{Q_{U^*}, -\Phi\}, -\Phi\} = 0. \tag{22}$$

Lemma 2.13. *Let (U, Q_U, Q_{U^*}) and (V, Q_V, Q_{V^*}) be finite dimensional Lie bialgebras. The following quadruple forms a V-data:*

- *the graded Lie algebra*

$$L := C_{(\geq 1,\geq 1)}\Big(T^*[2](U \times V)[1]\Big)[2] = \Big(\wedge^{\geq 1}(U^* \times V^*) \otimes \wedge^{\geq 1}(U \times V)\Big)[2]$$

- *its abelian subalgebra $\mathfrak{a} := (\wedge^{\geq 1}U^* \otimes \wedge^{\geq 1}V)[2]$*

- *the natural projection $P\colon L \to \mathfrak{a}$ with kernel*

$$ker(P) = \Big(\wedge U^* \otimes \wedge^{\geq 1}V^* \otimes \wedge^{\geq 1}(U \times V)\Big)[2] + \Big(\wedge^{\geq 1}(U^* \times V^*) \otimes \wedge^{\geq 1}U \otimes \wedge V\Big)[2]$$

- *$\Delta := Q_U + Q_{U^*} + Q_V - Q_{V^*}$,*

hence by Thm. 1 we obtain a $L_\infty[1]$-structure $\mathfrak{a}^P_\Delta$.

$\Phi \in \mathfrak{a}_0$ is a Maurer-Cartan element in $\mathfrak{a}^P_\Delta$ iff ϕ is a Lie bialgebra morphism.

Proof. Since $T^*[2](U \times V)[1]$ is endowed with a Poisson bracket of bidegree $(-1,-1)$, the shifted space of functions $C\big(T^*[2](U \times V)[1]\big)[2]$ is a graded Lie algebra and L is a graded Lie subalgebra. $Ker(P)$ is a Lie subalgebra of L, as can be checked in coordinates. Clearly Δ lies in $ker(P)$, and

$$\{\Delta, \Delta\} = \{Q_U + Q_{U^*}, Q_U + Q_{U^*}\} + \{Q_V - Q_{V^*}, Q_V - Q_{V^*}\} = 0.$$

Hence the above quadruple forms a V-data, and we can apply Thm. 1.

To compute the Maurer-Cartan elements of $\mathfrak{a}^P_\Delta$, take $\Phi \in \mathfrak{a}_0 = U^* \otimes V$. One computes easily in coordinates that

$$P\{\Delta, \Phi\} = \{Q_U - Q_{V^*}, \Phi\}$$
$$P\{\{\Delta, \Phi\}, \Phi\} = \{\{Q_V + Q_{U^*}, \Phi\}, \Phi\}$$

while all other terms of the Maurer-Cartan equation vanish. Separating the terms in $\wedge^2 U^* \otimes V$ from those in $U^* \otimes \wedge^2 V$ we conclude that Φ is a Maurer-Cartan element of $\mathfrak{a}^P_\Delta$ iff the equations (21) and (22) are satisfied, which in turn is equivalent to ϕ being a a Lie bialgebra morphism. □

Lemma 2.13 allows us to apply Thm. 3 (and Rem. 1.8). Hence we deduce:

Corollary 2.14. *Let (U, Q_U, Q_{U^*}) and (V, Q_V, Q_{V^*}) be finite dimensional Lie bialgebras and $\phi\colon U \to V$ a Lie bialgebra morphism. Let $(L, \mathfrak{a}, P, \Delta)$ be as in Lemma 2.13.*

1) Let $\tilde{\phi}\colon U \to V$ a linear map. Then

$$\phi + \tilde{\phi} \text{ is a Lie bialgebra morphism} \quad \Leftrightarrow \quad \tilde{\Phi} \text{ is a MC element of } \mathfrak{a}^{P_\Phi}_{\Delta}.$$

2) For all $\tilde{Q}_U \in C_{(2,1)}(T^*[2]U[1])$ and $\tilde{Q}_{U^*} \in C_{(1,2)}(T^*[2]U[1])$, for all analogously defined $\tilde{Q}_V, \tilde{Q}_{V^*}$, and for all linear maps $\tilde{\phi}\colon U \to V$:

$$\begin{cases} (U, Q_U + \tilde{Q}_U, Q_{U^*} + \tilde{Q}_{U^*}) \text{ and } (V, Q_V + \tilde{Q}_V, Q_{V^*} + \tilde{Q}_{V^*}) \text{ are Lie bialgebras} \\ \phi + \tilde{\phi} \text{ is a Lie bialgebra morphism between these new Lie bialgebra structures} \end{cases}$$

$$\Leftrightarrow \; ((\tilde{Q}_U + \tilde{Q}_{U^*} + \tilde{Q}_V - \tilde{Q}_{V^*})[1], \tilde{\Phi}) \text{ is a MC element of } (L[1] \oplus \mathfrak{a})^{P_\Phi}_{\Delta}.$$

Remark 2.15. We check that the V-data appearing in Cor. 2.14 is filtered. We have a direct sum decomposition $L = \oplus_{k \geq -1} L^k$ where $L^k := L \cap \oplus_{q+r=k+1} \big(\wedge^q U^* \otimes \wedge V^* \otimes \wedge U \otimes \wedge^r V\big)[2]$. In other words, L^k is spanned by monomials in L whose total number of u's and $\frac{\partial}{\partial v}$'s, in coordinates, is exactly $k+1$. Then $\mathcal{F}^n L := \oplus_{k \geq n} L^k$ is a complete filtration of the vector space L. One checks easily that $(L, \mathfrak{a}, P, \Delta)$ is a filtered V-data.

Remark 2.16. It seems that there is no way to recover Cor. 2.14 simply applying the results of Cor. 2.3 twice (once to Lie algebra morphism $\phi\colon U \to V$ and once to the Lie algebra morphism $\phi^*\colon V^* \to U^*$), since the latter procedure would deform ϕ and ϕ^* to two Lie algebra morphisms $\alpha\colon U \to V$ and $\beta\colon V^* \to U^*$ which are not necessarily duals of each other.

2.4 Maurer-Cartan elements of L_∞-algebra structures

Fix a (possibly infinite dimensional) graded vector space W. We show that the space of pairs

$$(L_\infty[1]\text{-algebra structures on } W, \text{ Maurer-Cartan elements for this structure})$$

is governed by a Maurer-Cartan equation. We will ignore all convergence issues in this subsection; they are automatically dealt with if one works formally, see Lemma 1.13.

We refer to §A.2 for the background material on coderivations. In §A.3 we recall that $L_\infty[1]$-algebra structures on W are in bijection with degree 1 self-commuting coderivations Θ on $\overline{SW} := \oplus_{k=1}^{\infty} S^k W$, we show that there is a canonical embedding $\alpha\colon W \hookrightarrow Coder(SW)$, and that there is a canonical bracket-preserving embedding $\mathcal{J}\colon Coder(\overline{SW}) \hookrightarrow Coder(SW)$ whose image annihilates $1 \in SW$. In §A.3 we further prove that all $L_\infty[1]$-algebra structures are obtained by the derived bracket construction:

Proposition 2.17. *Let W be an L_∞[1]-algebra, and Θ the corresponding coderivation of $\overline{SW}$. The following quadruple forms a V-data:*

- *the graded Lie algebra $L := Coder(SW)$*

- *its abelian subalgebra* $\mathfrak{a} := \{\alpha_w : w \in W\}$
- *the projection* $P\colon L \to \mathfrak{a}\ ,\ \tau \mapsto \alpha_{\tau(1)}$
- $\Delta := \mathcal{J}\Theta$.

The induced $L_\infty[1]$*-structure on* $\mathfrak{a}$ *given by Thm. 1 is exactly the original* L_∞*[1]-structure on* W*, under the canonical identification* $W \cong \mathfrak{a}, w \mapsto \alpha_w$*.*

We apply Cor. 1.9, choosing $\Theta = 0$ above and restrict to $\{\tau \in Coder(SW) : \tau(1) = 0\} = Ker(P) \subset L$ (see Rem. 1.7). We obtain:

Corollary 2.18. $\{\tau \in Coder(SW) : \tau(1) = 0\}[1] \oplus W$*, endowed with the* $L_\infty[1]$*-algebra structure specified in Cor. 1.9, has the following property: for all* $\tilde{\Theta} \in Coder(\overline{SW})_1$ *and* $\tilde{\Phi} \in W_0$*:*

$$\begin{cases} \tilde{\Theta} \textit{ defines an } L_\infty[1]\textit{-algebra structure on } W \\ \tilde{\Phi} \textit{ is a MC element of this } L_\infty[1]\textit{-algebra structure on } W \end{cases}$$
$$\Leftrightarrow\ (\mathcal{J}\tilde{\Theta}[1], \tilde{\Phi}) \textit{ is a MC element of } \{\tau \in Coder(SW) : \tau(1) = 0\}[1] \oplus W$$

One can show that the image of the embedding $\mathcal{J}$ is exactly $\{\tau \in Coder(SW) : \tau(1) = 0\}$, so Cor. 2.18 is a statement about *all* Maurer-Cartan elements of $\{\tau \in Coder(SW) : \tau(1) = 0\}[1] \oplus W$.

2.5 L_∞-algebra morphisms

We consider deformations of a pair of arbitrary $L_\infty[1]$-algebras and of a $L_\infty[1]$-morphism between them. We show that deformations of the morphism with fixed $L_\infty[1]$-algebra structures are ruled by a $L_\infty[1]$-algebra (this follows also from Shoikhet's work, see [30, §3][14]), and then show that there is an $L_\infty[1]$-algebra governing arbitrary deformations.

We will use the following notation. When E and F are two vector spaces, we will denote by $L(E,F)$ the set of linear maps from E to F and use $L(E) := L(E,F)$ when $E = F$.

Let U and V be two graded vector spaces. Denote $\overline{S(U \oplus V)} := \oplus_{k\geq 1} S^k(U \oplus V)$. Let

$$L := L\left(\overline{S(U \oplus V)}, U \oplus V\right) = \prod_{i\geq 1} \bigoplus_{q+r=i} L_U^{q,r} \oplus L_V^{q,r}, \tag{23}$$

where

$$L_U^{q,r} := \left\{\Pi_U \circ l \circ \Pi^{q,r} : l \in L(S^{q+r}(U \oplus V), U \oplus V)\right\}$$

for $\Pi^{q,r}\colon S^{q+r}(U \oplus V) \to S^q U \otimes S^r V$ and $\Pi_U\colon U \oplus V \to U$ the canonical projections. Consider the subspace

$$\mathfrak{a} := \prod_{q\geq 1} L_V^{q,0} \cong L(\overline{SU}, V).$$

Thanks to the decomposition (23) one has a projection $P\colon L \to \mathfrak{a}$. Notice that the vector space L has a natural $\mathbb{Z}$-grading: $L = \oplus_{n\in\mathbb{Z}} L_n$, where a map $l\colon \overline{S(U \oplus V)} \to U \oplus V$ lies in L_n if it raises the degree by n.

As remarked by Stasheff [31], L is a graded Lie algebra: the isomorphism of graded vector spaces

$$L \cong Coder(\overline{S(U \oplus V)}) \tag{24}$$

given in Proposition A.8 allows to define the Lie bracket on L, the *Gerstenhaber bracket*, as the pullback of the graded commutator of coderivations.

Proposition 2.19. *Let U and V be two graded vector spaces equipped with $L_\infty[1]$-algebra structures $\mu = (\mu_i)_{i\geq 1}$ and $\nu = (\nu_j)_{j\geq 1}$, where $\mu_i \in L_U^{i,0}$ and $\nu_j \in L_V^{0,j}$. The following quadruple (with the previous notations) forms a V-data:*

- *the graded Lie algebra L,*
- *its abelian subalgebra $\mathfrak{a}$,*
- *the projection $P\colon L \to \mathfrak{a}$,*
- *$\Delta := \mu + \nu$.*

Proof. The proof, which uses Lemma A.9, is analogous to the proof of Lemma 3.1 and is therefore left to the reader. □

Proposition 2.20. *$\Phi \in MC(\mathfrak{a}_\Delta^P) \Leftrightarrow \Phi$ is a morphism of $L_\infty[1]$-algebras.*

Proof. Fix $\Phi \in \mathfrak{a}_0$. Our aim is to show that the condition for Φ to be a Maurer-Cartan element for the $L_\infty[1]$-algebra $\mathfrak{a}_\Delta^P$ (see Remark 1.6),

$$Pe^{[-,\Phi]}(\mu + \nu) = 0,$$

is equivalent to the condition for Φ to be a morphism of $L_\infty[1]$-algebras, i.e., for all $s \geq 1$ and $u_1, \dots, u_s \in U$:

$$\sum_{I \amalg J = [s]} \Phi_{|J|+1}(\mu_{|I|}(U_I) \cdot U_J) = \sum_{n=1}^{s} \frac{1}{n!} \sum_{I_1 \amalg \cdots \amalg I_n = [s]} \nu_n(\Phi_{|I_1|}(U_{I_1}) \dots \Phi_{|I_n|}(U_{I_n})), \tag{25}$$

where $[s] := \{1, \dots, s\}$, $\amalg$ means disjoint union and $U_I = u_{\alpha_1} \dots u_{\alpha_j}$ when $I = \{\alpha_1, \dots, \alpha_j\}$. Some of the I_i's in the expression $I_1 \amalg \cdots \amalg I_n = [s]$ can be empty. One will use the convention that $\Phi_{|\emptyset|}(U_\emptyset) = 0$ and $U_I \cdot U_\emptyset = U_I$. Here we decompose Φ as a sum of its homogeneous elements with respect to the polynomial degree, i.e. $\Phi = \sum \Phi_n$ where $\Phi_n \in L_V^{n,0}$.

It will be convenient to use the isomorphism (24) to view the elements of L as coderivations, because in this case the Lie bracket is the graded commutator. The coderivation corresponding to Φ (resp. to μ, ν) will be denoted by $\bar{\Phi}$ (resp. $\bar{\mu}$, $\bar{\nu}$).

Φ is a Maurer-Cartan element of the $L_\infty[1]$-algebra $\mathfrak{a}_\Delta^P$ iff

$$Pe^{[-,\bar{\Phi}]}(\bar{\mu} + \bar{\nu}) = 0.$$

But, with the notation $ad_\Phi := [-, \Phi]$, one has

$$e^{[-,\bar{\Phi}]} = \sum_{n \geq 0} \frac{1}{n!} {ad_{\bar{\Phi}}}^n,$$

and one can compute ${ad_{\bar{\Phi}}}^n(\bar{\mu})$ and ${ad_{\bar{\Phi}}}^n(\bar{\nu})$ with the expansion

$$ {ad_{\bar{\Phi}}}^n(\tau) = \sum_{k+l=n} (-1)^k \binom{n}{k} \bar{\Phi}^k \tau \bar{\Phi}^l. $$

Therefore everything boils down to compute terms of the form

$$ \bar{\Phi}^k \tau \bar{\Phi}^l (u_1 \dots u_s). $$

The results of these computations for $\tau = \bar{\nu}$ and $\tau = \bar{\mu}$ with $n = k + l$ are claims 1 and 2 respectively, and give the two sides of the equation (25).

Claim 1. *The term*

$$ pr_V(\bar{\Phi}^k \circ \bar{\nu} \circ \bar{\Phi}^l(U_{[s]})) $$

always vanishes except for $l = n$ for which one has

$$ pr_V(\bar{\Phi}^0 \circ \bar{\nu} \circ \bar{\Phi}^n(U_{[s]})) = \sum_{I_1 \amalg \dots \amalg I_n = [s]} \bar{\nu}_n(\Phi_{|I_1|}(U_{I_1}) \dots \Phi_{|I_n|}(U_{I_n})). $$

Claim 2. *The term*

$$ pr_V(\bar{\Phi}^k \circ \bar{\mu} \circ \bar{\Phi}^l(U_{[s]})) $$

always vanishes, except for $k = n = 1$ for which one has

$$ pr_V(\bar{\Phi}^1 \circ \bar{\mu}(U_{[s]})) = \sum_{I \amalg J = [s]} \Phi_{|J|+1}(\mu_{|I|}(U_I) \cdot U_J). $$

Combining the results of claims 1 and 2 finishes the proof of Proposition 2.20. □

We now state a lemma and use it to prove claims 1 and 2. All along we fix $s \geq 1$ and $u_1, \dots, u_s \in U$.

Lemma 2.21. *For all $t \geq 0$*

$$ \bar{\Phi}^t(U_{[s]}) = \sum_{I_1 \amalg \dots \amalg I_{t+1} = [s]} \Phi_{|I_1|}(U_{I_1}) \dots \Phi_{|I_t|}(U_{I_t}) \cdot U_{I_{t+1}}. \tag{26} $$

Proof. Apply formula (39) t times and remark that since Φ admits only elements in U, terms of the form $\Phi(\Phi(U_I) \cdot U_{I'})$ can not appear in the obtained expression. The case $t = 0$ is a convention. □

Proof of claim 1. We apply the formula (39) to $\bar{\nu}$ evaluated on the right hand side of the equation (26), with $t = l$ to get

$$ \sum \nu_{|J|}(\Phi_{|I_{\alpha_1}|}(U_{I_{\alpha_1}}) \dots \Phi_{|I_{\alpha_j}|}(U_{I_{\alpha_j}}) \cdot U_{I_{l+1}}) \cdot \Phi_{|I_{\beta_1}|}(U_{I_{\beta_1}}) \dots \Phi_{|I_{\beta_k}|}(U_{I_{\beta_k}}) \cdot U_{I_{l+2}}, $$

where $\{\alpha_1, \dots, \alpha_j\} = J$ and $\{\beta_1, \dots, \beta_k\} = K$, and the sum is over $I_1 \amalg \dots \amalg I_{l+2} = [s]$ and $J \amalg K = [l]$.

Now, since ν admits only elements in U, the term $U_{I_{l+1}}$ must be absent in the previous expression, i.e. one has

$$\bar{\nu} \circ \bar{\Phi}^l(U_{[s]}) = \sum \nu_{|J|}(\Phi_{|I_{\alpha_1}|}(U_{I_{\alpha_1}}) \dots \Phi_{|I_{\alpha_j}|}(U_{I_{\alpha_j}})) \cdot \Phi_{|I_{\beta_1}|}(U_{I_{\beta_1}}) \dots \Phi_{|I_{\beta_k}|}(U_{I_{\beta_k}}) \cdot U_{I_{l+1}},$$

(sum over $I_1 \amalg \dots \amalg I_{l+1} = [s], J \amalg K = [l]$).

We are interested in evaluating the expression $\bar{\Phi}^k \circ \bar{\nu} \circ \bar{\Phi}^l(U_{[s]})$, with $k+l=n$. By applying Lemma 2.21 with $t=k$ to the last expression, and by the fact that Φ admits only terms in U, one gets

$$\bar{\Phi}^k \circ \bar{\nu} \circ \bar{\Phi}^l(U_{[s]}) = \sum \nu_{|J|}(\Phi_{|I_{\alpha_1}|}(U_{I_{\alpha_1}}) \dots \Phi_{|I_{\alpha_j}|}(U_{I_{\alpha_j}})) \cdot \Phi_{|I_{\beta_1}|}(U_{I_{\beta_1}}) \dots \Phi_{|I_{\beta_k}|}(U_{I_{\beta_k}}) \cdot U_{I_{n+1}}.$$

(sum over $I_1 \amalg \dots \amalg I_{n+1} = [s]; J \amalg K = [n]$).

Finally, if one considers the terms in the above formula which belong to V, one has

$$pr_V(\bar{\Phi}^k \circ \bar{\nu} \circ \bar{\Phi}^l(U_{[s]})) = \sum_{I_1 \amalg \dots \amalg I_n = [s]} \nu_n(\Phi_{|I_1|}(U_{I_1}) \dots \Phi_{|I_n|}(U_{I_n})).$$

□

Proof of claim 2. We apply the formula (39) to $\bar{\mu}$ evaluated on the right hand side of the equation (26), with $t=l$ and remark that since μ admits only elements in U, terms of the form $\mu(\Phi(U_I) \cdot U_{I'})$ can not appear in the obtained expression. Therefore one has

$$\bar{\mu} \circ \bar{\Phi}^l(U_{[s]}) = \sum_{I_1 \amalg \dots \amalg I_{l+2} = [s]} \Phi_{|I_1|}(U_{I_1}) \dots \Phi_{|I_l|}(U_{I_l}) \cdot \mu_{|I_{l+1}|}(U_{I_{l+1}}) \cdot U_{I_{l+2}}.$$

We now evaluate $\bar{\Phi}^k \circ \bar{\mu} \circ \bar{\Phi}^l(U_{[s]})$ by applying Lemma 2.21 to the previous expression, with $t=k$. Since Φ admits only elements in U, terms of the form $\Phi(\Phi(U_I) \cdot U_{I'})$ can not appear in the obtained expression. Hence one gets (remember that $n=k+l$)

$$\sum_{I_1 \amalg \dots \amalg I_{n+2} = [s]} \Phi_{|I_1|}(U_{I_1}) \dots \Phi_{|I_n|}(U_{I_n}) \cdot \mu_{|I_{n+1}|}(U_{I_{n+1}}) \cdot U_{I_{n+2}}$$
$$+ \sum_{I_1 \amalg \dots \amalg I_{n+2} = [s]} \Phi_{|I_1|}(U_{I_1}) \dots \Phi_{|I_n|+1}(U_{I_n} \cdot \mu_{|I_{n+1}|}(U_{I_n+1})) \cdot U_{I_{n+2}}.$$

In the previous expression, there are terms which belong to V only if n=k=1. In this case one has

$$pr_V(\bar{\Phi} \circ \bar{\mu}(U_{[s]})) = \sum_{I \amalg J = [s]} \Phi_{|J|+1}(\mu_{|I|}(U_I) \cdot U_J).$$

□

Prop. 2.19 and Prop. 2.20 allow us to apply Thm. 3 (and Rem. 1.8) and deduce:

Corollary 2.22. *Let U, V be $L_\infty[1]$-algebras and $\Phi \in L(\overline{SU}, V)$ a $L_\infty[1]$-morphism from U to V and let $(L, \mathfrak{a}, P, \Delta)$ as in Prop. 2.19.*

1) Let $\tilde{\Phi} \in L_0(\overline{SU}, V) = \mathfrak{a}_0$*. Then*

$$\Phi + \tilde{\Phi} \text{ is an } L_\infty[1]\text{-morphism} \quad \Leftrightarrow \quad \tilde{\Phi} \in MC(\mathfrak{a}_\Delta^{P_\Phi}).$$

2) For all degree one coderivations $\tilde{Q}_U$ *on* $\overline{SU}$ *and* $\tilde{Q}_V$ *on* $\overline{SV}$ *and for all* $\tilde{\Phi} \in L_0(\overline{SU}, V)$*:*

$$\begin{cases} Q_U + \tilde{Q}_U \text{ and } Q_V + \tilde{Q}_V \text{ define } L_\infty[1]\text{-algebra structures on } U, V \\ \Phi + \tilde{\Phi} \text{ is a } L_\infty[1]\text{-morphism between these } L_\infty[1]\text{-algebra structures} \end{cases}$$
$$\Leftrightarrow \ ((\tilde{Q}_U + \tilde{Q}_V)[1], \tilde{\Phi}) \in MC((L[1] \oplus \mathfrak{a})_\Delta^{P_\Phi})$$

Remark 2.23. We have a direct product decomposition $L = \prod_{k \geq -1} L^k$ where $L^k := L_U^{k+1,\bullet} \oplus L_V^{k,\bullet}$. Here we use the short-hand notation $L_V^{k,\bullet} := \prod_{r \geq 0} L_V^{k,r}$. Then $\mathcal{F}^n L := \prod_{k \geq n} L^k$ is a complete filtration of the vector space L. One checks easily that $(L, \mathfrak{a}, P, \Delta)$ is filtered V-data (Def. 1.11).

3 Applications to associative algebras

In this section we treat the case of a morphism between two associative algebras. The cohomology theory governing infinitesimal simultaneous deformations of two associative algebras and a morphism between them has been defined in the context of cohomology of diagrams by M. Gerstenhaber and S.D. Schack in [11]. However was still open the problem of writing the whole deformation equation as a Maurer-Cartan equation. The first author, Markl and Yau in [7] exhibited a L_∞-algebra which enabled to write this deformation equation as a Maurer-Cartan equation. This was based on operadic techniques. Note that similar results were also obtained via simplicial methods and model categories in [3]. However, it has not been proven yet that the L_∞ algebras of [3] and [7] are quasi-isomorphic. We show in this section how we can recover the results of [7] by means of our Thm. 3, which requires much less technology.

3.1 Morphisms of associative algebras

We will use some notations introduced in §2.5. Moreover, if E and F are two vector spaces, for any $n \geq 1$ and $I \amalg J = [n] := \{1, \ldots, n\}$, consider the notation

$$T^{I,J}(E,F) := \{x_1 \otimes \cdots \otimes x_n \in T^n(E \oplus F) : x_k \in E \text{ when } k \in I, x_k \in F \text{ otherwise } \}.$$

One has the decomposition

$$T^n(E \oplus F) = \bigoplus_{I \amalg J = [n]} T^{I,J}(E,F),$$

and therefore one can consider the projection $\Pi^{I,J}$ onto $T^{I,J}(E,F)$. One considers also the canonical projection Π_E (resp Π_F) from $E \oplus F$ onto E (resp F).

One will denote the set of n-linear maps from E to F by $L^n(E,F) := L(T^n E, F)$ and by $L^n(E) := L^n(E,E)$ when $E = F$. One has the decomposition:

$$L^n(E \oplus F) = \bigoplus_{I \amalg J = [n]} L_E^{I,J} \oplus L_F^{I,J}, \tag{27}$$

for $L_E^{I,J} := \{\Pi_E \circ l \circ \Pi^{I,J} : l \in L^n(E \oplus F)\}$. The decomposition (27) defines a projection

$$P \colon \prod\nolimits_{n \geq 1} L^n(E \oplus F) \to \oplus_{n \geq 1} L_F^{[n],\emptyset}.$$

Consider a morphism $\Phi \colon U \to V$ between two associative algebras (U, μ) and (V, ν), apply the above notations to $E := U[1]$ and $F := V[1]$, and consider μ and ν as elements of $L^2(U[1])$ and $L^2(V[1])$. As noticed by Stasheff in [31], the canonical identification $\prod_{n \geq 1} L^n(E \oplus F) \cong Coder(\overline{T(E \times F)})$ of Prop. A.8 makes $\prod_{n \geq 1} L^n(E \oplus F)$ into a graded Lie algebra, whose bracket is called Gerstenhaber bracket.

Lemma 3.1. *The following quadruple forms a V-data:*

- *the graded Lie algebra* $L := \prod_{n \geq 0} L_n$ *with* $L_n := L^{n+1}((U \oplus V)[1])$ *with Gerstenhaber bracket* $[\cdot, \cdot]$
- *its abelian subalgebra* $\mathfrak{a} = \prod_{n \geq 0} \mathfrak{a}_n$ *with* $\mathfrak{a}_n := L_{V[1]}^{[n+1],\emptyset} \cong L(T^{n+1}U[1], V[1])$
- *the natural projection* $P \colon L \to \mathfrak{a}$ *given above*
- $\Delta := \mu + \nu$

hence by Thm. 1 we obtain a $L_\infty[1]$*-structure* $\mathfrak{a}_\Delta^P$.

$\Phi \in \mathfrak{a}_0$ *is a Maurer-Cartan element in* $\mathfrak{a}_\Delta^P$ *iff* Φ *is a morphism of associative algebras between* μ *and* ν.

Proof. To see that $\mathfrak{a}$ is an abelian graded Lie subalgebra of L, remark that elements of $\mathfrak{a}$ are maps which produce vectors in V and accept only terms in U. Therefore their composition is zero.

Next we show that $KerP$ is a graded Lie subalgebra of L. To this aim use the decomposition $KerP = A \oplus B$ where

$$A_n = \bigoplus_{I \amalg J = [n], |J| > 0} L_{V[1]}^{I,J},$$
$$B_n = \bigoplus_{I \amalg J = [n]} L_{U[1]}^{I,J}.$$

Let $\alpha, \alpha' \in A, \beta \in B$ and $\gamma \in KerP$. One has $\alpha \circ \beta, \alpha \circ \alpha' \in A$ and $\beta \circ \gamma \in B$, showing that $KerP = A \oplus B$ is closed under the Gerstenhaber bracket. Further since $\nu \in A$ and $\mu \in B$, one has $\Delta \in KerP$.

Last we show that $[\Delta, \Delta] = 0$. Indeed,

$$[\Delta, \Delta] = [\mu, \mu] + [\nu, \nu] + 2[\mu, \nu].$$

Since μ and ν are associative algebras, by Prop. A.11, $[\mu,\mu]$ and $[\nu,\nu]$ vanish. Now, by definition of the bracket, $[\mu,\nu] = \mu\circ(\nu\otimes Id - Id\otimes\nu) - \nu\circ(\mu\otimes Id - Id\otimes\mu)$ but μ accepts only terms in V, whereas ν produces elements in U, hence the first summand of the right hand side vanishes. Similarly for the second summand. This concludes the proof that $(L,\mathfrak{a},P,\Delta)$ forms a V-data.

Fix $\Phi\in\mathfrak{a}_0 = L(U[1],V[1])$. It will be convenient to view the elements of L as coderivations, because in this case the Lie bracket is the graded commutator. The coderivation corresponding to Φ (Proposition A.8) will be denoted by $\bar{\Phi}$. It is characterized by its only non vanishing corestriction, which is $\bar{\Phi}^1_1(u+v)=\Phi(u)$ where $u\in U$ and $v\in V$.

By Remark 1.6, Φ is a Maurer-Cartan element of the $L_\infty[1]$-algebra $\mathfrak{a}^P_\Delta$ iff

$$Pe^{[-,\bar{\Phi}]}(\mu+\nu)=0. \tag{28}$$

Since

$$e^{[-,\bar{\Phi}]}=\sum_{n\geq 0}\frac{1}{n!}{ad_{\bar{\Phi}}}^n,$$

writing $ad_\Phi := [-,\Phi]$, we compute ${ad_{\bar{\Phi}}}^n(\mu)$ and ${ad_{\bar{\Phi}}}^n(\nu)$ with the expansion

$${ad_{\bar{\Phi}}}^n(\tau)=\sum_{k=0}^n(-1)^k\bar{\Phi}^k\tau\bar{\Phi}^{n-k}.$$

Let us first remark that the commutator of a linear coderivation and a quadratic coderivation gives a quadratic coderivation. In particular ${ad_{\bar{\Phi}}}^n(\nu)$ and ${ad_{\bar{\Phi}}}^m(\mu)$ are quadratic coderivations and hence are only determined by their second Taylor coefficient, i.e. by their restriction to elements of $T^2(U\oplus V)$.

One observes that for elements x_1,x_2 in U (for elements in V, the expression would vanish),

$$\bar{\Phi}^2(x_1\otimes x_2)=\bar{\Phi}(\Phi(x_1)\otimes x_2+x_1\otimes\Phi(x_2))=2\Phi(x_1)\otimes\Phi(x_2)$$

lies in $T^2(V)$. Therefore, Φ can not be applied anymore, meaning that $\bar{\Phi}^n(x_1\otimes x_2)=0$ for all $n>2$. For the same reason, if τ has only quadratic Taylor coefficients, one has necessary $\bar{\Phi}^n\tau_{|T^2(U\oplus V)}=0$ for $n>1$, and even $\bar{\Phi}\tau_{|T^2(U\oplus V)}=0$ when the quadratic Taylor coefficients of τ have values in V. All these remark imply that the only non-vanishing ${ad_{\bar{\Phi}}}^n(\nu)_{|T^2(U\oplus V)}$ occurs for $n=2$:

$${ad_{\bar{\Phi}}}^2(\nu)(x_1\otimes x_2)=2\nu(\Phi(x_1)\otimes\Phi(x_2))$$

and the only non-vanishing ${ad_{\bar{\Phi}}}^n(\mu)_{|T^2(U\oplus V)}$ occurs for $n=1$:

$$ad_{\bar{\Phi}}(\mu)(x_1\otimes x_2)=-\Phi(\mu(x_1\otimes x_2).$$

Since μ and ν commute, we obtain that the l.h.s. of eq. (28) is

$$Pe^{[\mu+\nu,\bar{\Phi}]}(x_1\otimes x_2)=\nu(\Phi(x_1)\otimes\Phi(x_2))-\Phi(\mu(x_1\otimes x_2).$$

Hence $\bar{\Phi}$ satisfies eq. (28) iff Φ is a morphism of associative algebras. $\square$

To establish the connection with the problem of simultaneous deformations of morphisms and associative algebras, one considers the graded Lie subalgebra L' of L defined by

$$L'_i = L^{i+1}(U[1]) \;\oplus\; L^{i+1}(V[1]).$$

Thm. 3 and Remark 1.7 (which applies to L' since it contains Δ) imply:

Corollary 3.2. *Let (U,μ) and (V,ν) be associative algebras and $\Phi\colon U \to V$ a morphism of associative algebras. Let $(L,\mathfrak{a},P,\Delta)$ as in Lemma 3.1 and L' as above.*

For all $\tilde{\mu}+\tilde{\nu} \in L'_1$, and for all linear maps $\tilde{\Phi}\colon U \to V$:

$$\begin{cases} \mu+\tilde{\mu} \textit{ and } \nu+\tilde{\nu} \textit{ define associative algebra structures on } U \textit{ and } V \\ \Phi+\tilde{\Phi} \textit{ is an associative algebra morphism between these new associative algebra structures} \end{cases}$$

$$\Leftrightarrow ((\tilde{\mu}+\tilde{\nu})[1],\tilde{\Phi}) \in MC((L'[1]\oplus\mathfrak{a})^{P_\Phi}_{\Delta}).$$

Remark 3.3. Analogously to Remark 2.4, we have a direct product decomposition $L = \prod_{k\geq -1}\mathcal{L}^k$ where $\mathcal{L}^k := \bigoplus_{|I|=k+1} L^{I,\bullet}_{U[1]} \oplus \bigoplus_{|I|=k} \mathcal{L}^{I,\bullet}_{V[1]}$. Then $\mathcal{F}^n L := \prod_{k\geq n} L^k$ is a complete filtration of the vector space L. One checks easily that $(L,\mathfrak{a},P,\Delta)$ is filtered V-data (Def. 1.11).

3.1.1 Explicit expressions for the multibrackets

We now write out explicitly the multi-brackets of $(L'[1]\oplus\mathfrak{a})^{P_\Phi}_{\Delta}$, the $L_\infty[1]$-algebra which by Cor. 3.2 controls the deformations of associative algebra and their morphisms.

Let us denote by $\mathcal{P}_m[n]$ the set of ordered m-tuples of distinct points in $\{1,\ldots,n+1\}$. For any $I \in \mathcal{P}_m[n]$ we will denote by $x_V \circ_I (a_1,\ldots,a_n)$, the element obtained by plugging a_i into the I_i-th input of x_V, and by $x_V \circ_{I,\Phi} (a_1,\ldots,a_n)$ the element obtained by further plugging Φ in the $n+1-m$ remaining inputs of x_V. Similarly, $a \circ_i \mu$ will mean the composition of a by μ at its i-th input. We will also use the notations

$$da = \nu(a\otimes\Phi) + \nu(\Phi\otimes a) - (-1)^n \sum_{i=1}^{n} a\circ_i \mu$$

and $d^\mu x_U = [\mu, x_U]$. With these notations, explicit formulas are given by:

Proposition 3.4. *Let (U,μ) and (V,ν) be associative algebras and $\Phi\colon U\to V$ a morphism of associative algebras, and adopt the notation of Corollary 3.2. The $L_\infty[1]$-multi-brackets of $(L'[1]\oplus\mathfrak{a})^{P_\Phi}_{\Delta}$ are given as follows:*

Given $(x[1],a) \in (L'[1]\oplus\mathfrak{a})_n$, i.e $x = (x_U,x_V) \in L^{n+1}(U[1]) \oplus L^{n+1}(V[1])$ and $a \in L^n(U[1],V[1])$, one has

$$d(x[1],a) = (-d^\mu x_U - d^\nu x_V, -\Phi\circ x_U + x_V \circ \Phi^{\otimes^n} + da) \tag{29}$$

and

$$\{x,a\} = \sum_{i\in[n+1]} x_V \circ_{i,\Phi} a - (-1)^{|x||a|}\sum_j a\circ_j x_U. \tag{30}$$

If we moreover consider $a_1, \dots, a_m \in \mathfrak{a}$ (for $m \geq 2$), then one has

$$\{x, a_1, \dots, a_m\} = \sum_{I \in \mathcal{P}_m[n]} \epsilon(I) x_V \circ_{I,\Phi} (a_1, \dots, a_m) \tag{31}$$

$$\{a_1, a_2\} = \mu(a_1 \otimes a_2). \tag{32}$$

Remark 3.5. In [7] formulas were given for an L_∞-algebra governing simultaneous deformations of associative algebras and their morphisms. Those formulas agree with the formulas of Prop. 3.4 modulo signs, which come from the fact that here we only give the $L_\infty[1]$-algebra multibrackets. If one wants to recover the original formulas of [7], one needs to desuspend this $L_\infty[1]$-algebra as indicated in Remark 1.2.

Proof. We first prove (29). It suffices to explicit the expression (4) of Theorem 2, therefore we will determine (a) $-D(x)[1]$ and (b) $P_\Phi(x + Da)$, where $D = [\Delta, \cdot]$.

(a) Since μ and x_V can not be composed, $[\mu, x_V] = 0$ and hence $D(x_V) = [\nu, x_V]$. Since the similar result for x_U holds, one gets

$$-D(x)[1] = -d^\nu x_V - d^\mu x_U.$$

(b) Since a can not be composed on the right by ν and on the left by μ, one has $Da = \nu(a \otimes id) + \nu(id \otimes a) - (-1)^{|\mu||a|} \sum_{i=1}^n a \circ_i \mu$. In particular Da has only outputs in V, therefore Φ can only be right composed. Moreover it can only be right composed once since each of the summands of Da has at most one V input. Therefore $e^{[-,\Phi]} Da = Da + [Da, \Phi]$. After a look at the terms surviving the projection P, one gets

$$P_\Phi(Da) = \nu(a \otimes \Phi) + \nu(\Phi \otimes a) - (-1)^n \sum_{i=1}^n a \circ_i \mu.$$

One now remarks that in $e^{[-,\Phi]} x$, the only terms surviving the projection P are

$$P_\Phi x = -\Phi \circ x_U + x_V \circ \Phi^{\otimes^n},$$

therefore, identifying the terms in (4) gives (29).

We now prove (30). By definition of the Gerstenhaber bracket, one has

$$[x_U + x_V, a] = \sum_{i \in [n+1]} (x_U \circ_i a + x_V \circ_i a) - (-1)^{|x||a|} \sum_j (a \circ_j x_U + a \circ_j x_V).$$

But in this expression $x_U \circ_i a$ and $a \circ_i x_V$ vanish by incompatibility of the compositions. Now $P_\Phi(a \circ_j x_U) = a \circ_j x_U$ and $P_\Phi(x_V \circ_i a) = x_V \circ_{i,\Phi} a$, so one has proven (30), i.e.

$$\{x, a\} = \sum_{i \in [n+1]} x_V \circ_{i,\Phi} a - (-1)^{|x||a|} \sum_j a \circ_j x_U.$$

We now prove (31) for $m \geq 2$ by induction on m. Let us first start the induction by showing that

$$[[x, a_1], a_2] = \sum_{i,j} \epsilon(i,j) x_V \circ_I (a_1, a_2). \tag{33}$$

Let us remark that an element of L which has only U inputs and one V output can not be composed to the right or to the left by an element in $\mathfrak{a}$. This in particular applies to the element $[x_U, a_1]$, therefore one has $[[x_U, a_1], a_2] = 0$. Moreover, one has seen that

$$[x_V, a_1] = \sum_{i \in [n+1]} x_V \circ_i a_1.$$

But this term has one V output, therefore can not be left composed by a_2. This means, again by definition of the Gerstenhaber bracket, that one obtains eq. (33).

Let us now prove by induction that

$$[\dots[x, a_1], \dots, a_m] = \sum_{I \in \mathcal{P}_m[n]} \epsilon(I) x_V \circ_I (a_1, \dots, a_m).$$

We make the following observation (Obs): this element has a V output and therefore can not be composed to the left by an element in $\mathfrak{a}$. One has:

$$\begin{aligned} [[\dots[x, a_1], \dots, a_m], a_{m+1}] &= [\sum_{I \in \mathcal{P}_m[n]} \epsilon(I) x_V \circ_I (a_1, \dots, a_m)\ ,\ a_{m+1}] \\ &\overset{Obs}{=} \sum_{I \in \mathcal{P}_m[n]} \epsilon(I) \sum_{i \in I^c} (x_V \circ_I (a_1, \dots, a_m)) \circ_i a_{m+1} \\ &= \sum_{I \in \mathcal{P}_{m+1}[n]} \epsilon(I) x_V \circ_I (a_1, \dots, a_{m+1}), \end{aligned} \tag{34}$$

where in the first equality we used the induction step. It remains to apply the projection P_Φ. The above observation (Obs) applies in particular to the element Φ, therefore

$$e^{[-,\Phi]} x_V \circ_I (a_1, \dots, a_{m+1}) = \sum_{n \geq 0} \frac{1}{n!} x_V \circ_I (a_1, \dots, a_{m+1}) \circ \bar{\Phi}^n.$$

If one now compose this last equality with the projection P, one gets

$$P_\Phi(x_V \circ_I (a_1, \dots, a_{m+1})) = x_V \circ_{I,\Phi} (a_1, \dots, a_{m+1}).$$

Combining this last equality with (34) gives the result.

It remains to prove (32). But formula (6) is formally formula (5) with x replaced by Δ. Therefore one can compute the remaining brackets by replacing x by $\mu + \nu$ in formula (31). The only possibility is for $n = 2$, for which one gets (32). □

4 Applications to algebras over Koszul operads

The objective of this short section is to indicate how the techniques used in §2.5 and §3 work for other types of algebras. The theory of Koszul duality for operads (see [13] or [22]), provides for a type of algebra $\mathcal{P}$, i.e. for an operad $\mathcal{P}$ (for example for the operad $\mathcal{A}s$, encoding the type of associative algebras), a cooperad $\mathcal{P}^{\mathsf{i}}$. In this setting, given a graded vector space U, one can define $(\mathcal{P}^{\mathsf{i}}(U), \delta)$, the cofree coalgebra of type $\mathcal{P}^{\mathsf{i}}$ co-generated by U. Since it is cofree, one has the identification, as vector spaces:

$$\mathcal{P}^{\mathsf{i}}(U^*) \otimes U \simeq Coder(\mathcal{P}^{\mathsf{i}}(U)).$$

By Remark A.5, $Coder(\mathcal{P}^{\text{i}}(U))$ carries naturally the structure of a graded Lie algebra $[-,-]$, which can be pulled-back to $\mathcal{P}^{\text{i}}(U^*)\otimes U$. An *algebra μ of type $\mathcal{P}_\infty$*, or *$\mathcal{P}_\infty$-algebra* on the vector space U can then be defined as an element $\mu\in\mathcal{P}^{\text{i}}(U[1]^*)\otimes U[1]$ of internal degree 1 satisfying $[\mu,\mu]$=0. One can recover $\mathcal{P}$-algebras as the quadratic homogeneous $\mathcal{P}_\infty$-algebras.

4.1 Morphisms of $\mathcal{P}_\infty$-algebras

We are interested in deforming simultaneously two $\mathcal{P}_\infty$-algebras (U,μ) and (V,ν) and a morphism $\Phi\colon U\to V$ between them. The vector space $\mathcal{P}^{\text{i}}(V)$ carries a polynomial grading, and one considers

$$L^i:=\mathcal{P}^{\text{i}^i}((U[1]\oplus V[1])^*)\otimes(U[1]\oplus V[1]).$$

The graded Lie algebra $L:=\oplus_{i\geq 1}L^i$ admits an abelian subalgebra $\mathfrak{a}=\oplus_{i\geq 1}\mathfrak{a}^i$ with $\mathfrak{a}^i:=\mathcal{P}^{\text{i}^i}(U[1]^*)\otimes V[1]$. But one needs to work with the internal grading instead of the polynomial grading, and one will denote by $L:=\oplus_{i\geq 1}L_i$ and $\mathfrak{a}=\oplus_{i\geq 1}\mathfrak{a}_i$ their decompositions in homogeneous subspaces for the internal grading.

We believe that for any instance of Koszul operad $\mathcal{P}$, and any homotopy $\mathcal{P}$-algebras (U,μ) and (V,ν), the following Ansatz holds true.

Ansatz 4.1. *The following quadruple forms a filtered V-data (Def. 1.11):*

- *the graded Lie algebra $L:=\mathcal{P}^i((U\oplus V)[1]^*)\otimes(U\oplus V)[1]$ with bracket $[\cdot,\cdot]$*
- *its abelian subalgebra $\mathfrak{a}:=\mathcal{P}^i(U[1]^*)\otimes V[1]$*
- *the natural projection $P\colon L\to\mathfrak{a}$*
- *$\Delta:=\mu+\nu$.*

Further, denoting by $\mathfrak{a}^P_\Delta$ the $L_\infty[1]$-algebra obtained by Thm. 1:
$\Phi\in\mathfrak{a}_0$ lies in $MC(\mathfrak{a}^P_\Delta)$ iff Φ is a morphism of $\mathcal{P}_\infty$-algebras between μ and ν.

Applying Thm. 3 we obtain:

Corollary 4.2. *Let $\mathcal{P}$ be a Koszul operad, (U,μ) and (V,ν) be $\mathcal{P}_\infty$-algebras and $\Phi\colon U\to V$ be a morphism of $\mathcal{P}_\infty$-algebras. Assume that Ansatz 4.1 holds true for the corresponding V-data $(L,\mathfrak{a},P,\Delta)$ and that Φ defines an element of L. Let $L':=\mathcal{P}^i(U[1]^*)\otimes U[1]\oplus\mathcal{P}^i(V[1]^*)\otimes V[1]$.*

Then for all $\tilde\mu+\tilde\nu$ in ${L'}_1$, and for all $\tilde\Phi\in\mathfrak{a}_0$:

$$\begin{cases}\mu+\tilde\mu \text{ and } \nu+\tilde\nu \text{ define } \mathcal{P}_\infty\text{-algebra structures on } U \text{ and } V\\ \Phi+\tilde\Phi \text{ is an } \mathcal{P}_\infty\text{-algebra morphism between these new } \mathcal{P}_\infty\text{-algebra structures}\end{cases}$$

$\Leftrightarrow((\tilde\mu+\tilde\nu)[1],\tilde\Phi)$ *is a MC element of* $(L'[1]\oplus\mathfrak{a})^{P_\Phi}_\Delta$.

Let us illustrate this in the case $\mathcal{P}=\mathcal{A}s$. It is well known that $\mathcal{A}s^{\text{i}}=\mathcal{A}s$ and that the free coassociative coalgebra on a vector space U is given by the tensor coalgebra, therefore

$$\mathcal{P}^{\text{i}}(U^*)\otimes U=T(U^*)\otimes U.$$

So in particular, Proposition 3.1 is nothing else than Ansatz 4.1 for $\mathcal{P} = \mathcal{A}s$, with U and V graded vector spaces concentrated in degree 0. In particular μ and ν must be associative algebras and not arbitrary $\mathcal{A}_\infty$-algebras.

Another illustration is given if we take $\mathcal{P} = \mathcal{L}ie$, the Lie operad. One has $\mathcal{L}ie^{\text{¡}} = \mathcal{C}om$ (the cooperad of cocommutative coalgebras) and the free cocommutative coalgebra on a vector space U is given by the symmetric coalgebra, therefore

$$\mathcal{P}^{\text{¡}}(U^*) \otimes U = S(U^*) \otimes U.$$

This fact enables to recognize Proposition 2.19 as Ansatz 4.1 in disguise.

A Appendix

This appendix collects some background material on graded and formal geometry needed in the main text. Further, it presents the proof of Prop. 2.17.

Recall that a *graded vector space* is just a vector space W with a direct sum decomposition into subspaces $W = \oplus_{i\in\mathbb{Z}} W_i$. We refer to elements of W_i as "elements of degree i" and $|x|$ denotes the degree of x. The dual of W is naturally a graded vector space with $(W^*)_i = (W_{-i})^*$. For any integer k, $W[k]$ denotes the graded vector space with $(W[k])_i = W_{i+k}$. The set

$$L(E, E') := \{\text{linear maps from } E \text{ to } E'\}$$

is a graded vector space, with grading inherited from those of E and E': an element $\phi \in L(E, E')$ is said to be of degree k if it raises degrees by k, i.e. if $|\phi(x)| = |x| + k$ for all homogeneous $x \in E$. One denotes by $L(E, E')_k$ the set of linear maps of degree k, and $L(E, E') = \oplus_{k\in\mathbb{Z}} L(E, E')_k$. One easily checks that

$$L(E) := L(E, E)$$

is a graded Lie algebra when endowed with the graded commutator

$$[\phi, \psi] := \phi \circ \psi - (-1)^{|\phi||\psi|} \psi \circ \phi.$$

A.1 A primer on graded geometry: graded spaces and homological vector fields

We recall the notions of graded geometry needed in §2.1 - 2.3. See [29, §1.4] or [4] for more details.

Let W be a $\mathbb{Z}$-graded vector space. We introduce the symmetric algebra of W and its derivations.

- Let $TW := \mathbb{R} \oplus W \oplus W^{\otimes 2} \oplus \dots$ be the tensor algebra of W. It is a graded algebra, i.e., it is a graded vector space endowed with an associative morphism $TW \otimes TW \to TW$. Let SW be the quotient of TW by the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$, where x and y range over homogeneous elements of W. SW is a graded commutative algebra (see [29, §2, Def. 4.1], called the *graded symmetric algebra* of W.

- For any integer k, $Der(SW)_k$ denotes the space of degree k *derivations* of SW, i.e. $Q \in L(SW)_k$ which satisfy
$$Q(x \cdot y) = Q(x) \cdot y + (-1)^{k|x|} x \cdot Q(y).$$
$Der(SW) := \oplus_{k \in \mathbb{Z}} Der(SW)_k$ is closed under the graded commutator of linear endomorphisms, i.e. $Der(SW)$ is a Lie subalgebra of $(L(SW), [-,-])$.

Now let U be an n-dimensional, real vector space. Then $U[1]$ (resp. $(U[1])^*$) is a graded vector space concentrated in degree -1 (resp. 1). Exactly as ordinary vector spaces are instances of smooth manifolds, graded vector spaces are instances of graded manifolds. We do not give the definition of graded manifold here (see [4, §2.1]). Rather, we describe explicitly the two algebraic structures associated to the graded manifold $U[1]$ that will be used in this article:

- The space of "*functions* on $U[1]$"
$$C(U[1]) := S((U[1])^*).$$
It is a graded commutative algebra concentrated in degrees $0, \dots, n$. It is isomorphic to the ordinary exterior algebra $\wedge U^*$ of U^* (graded so that elements of $\wedge^k U^*$ have degree k).

- The space of "*vector fields* on $U[1]$"
$$\chi(U[1]) := Der(C(U[1])).$$
It is a graded Lie algebra, concentrated in degrees ≥ -1. As a graded vector space it is just $S((U[1])^*) \otimes U[1]$.

We give "coordinate expressions" for the above functions and vector fields. Notice that there is a canonical identification $\iota \colon U \to \chi_{-1}(U[1])$. An element $X \in U$ is identified with the vector field ι_X that satisfies $\iota_X(u) = \langle X, u \rangle$ for all $u \in (U[1])^* = C_1(U[1])$, where the pointy brackets denote the pairing of a vector space with its dual. (It is enough to specify how ι_X acts on $(U[1])^*$, since the latter generates the graded commutative algebra $C(U[1])$.)

- Choose a basis $X_1, \dots, X_n$ of U. The dual basis, viewed as a basis of $(U^*)[-1] = (U[1])^*$, will be denoted by
$$u_1, \dots, u_n.$$
We refer to the u_i as *coordinates on* $U[1]$. Notice that $|u_i| = 1$. The graded commutative algebra $C(U[1])$ is generated by the u_i, and a generic degree k element of $C(U[1])$ is a degree k polynomial expression in the u_i.

- $X_1, \dots, X_n$, under the identification of U with $\chi_{-1}(U[1])$, becomes a basis of $\chi_{-1}(U[1])$ which we denote by
$$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}.$$
Notice that $|\frac{\partial}{\partial u_i}| = -1$. We have $\frac{\partial}{\partial u_i}(u_j) = \delta_{ij}$. A general degree k element of $\chi(U[1])$ is of the form $\sum_{i=1}^n P_i \frac{\partial}{\partial u_i}$, where P_i is a degree $k+1$ polynomial expression on the u_j's.

Finally, by *homological vector field* on $U[1]$ we mean a degree 1 element $Q \in \chi(U[1])$ with the property that $[Q, Q] = 0$. Notice that a homological vector field is necessarily of the form $\sum c_{ij}^k u_i u_j \frac{\partial}{\partial u_k}$ for some constants c_{ij}^k.

A.2 A primer on formal geometry: coalgebras and homological coderivations

The notion of formal geometry is used in §2.4-2.5 and §3, and is dual to the notion of graded geometry. It is of use when one has to deal with infinite dimensional algebras. In this section we introduce the main objects of interest, homological coderivations. They are compact ways to handle algebras, or algebras up to homotopy: the brackets of these algebras are given by the Taylor coefficients of the corresponding coderivation. References for proofs can be found for example in [1], [5] or the appendix of [26].

Definition A.1. A *coalgebra* structure on a (possibly graded) vector space W consists of a (degree 0) linear map $\Delta : W \to W \otimes W$, called *coproduct*, satisfying the coassociativity condition

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta.$$

The only examples which will be of use here are:

Example A.2. If V is a (graded) vector space over the field $\mathbb{K}$, let us consider $TV = \oplus_{k=0}^{\infty} V^{\otimes^k}$ and $SV = \oplus_{k=0}^{\infty} S^k V$. They are coalgebras for the (degree-preserving) coproducts given respectively by

$$\Delta(x_1 \otimes \cdots \otimes x_n) := \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_{i-1}) \bigotimes (x_i \otimes \cdots \otimes x_n) \tag{35}$$

and

$$\Delta(x_1 \dots x_n) := \sum_{i=0}^{n} \sum_{\sigma \in Sh_{(i,n-i)}} \epsilon(\sigma; x_1, \dots, x_n) \cdot (x_{\sigma(1)} \dots x_{\sigma(i)}) \bigotimes (x_{\sigma(i+1)} \dots x_{\sigma(n)}). \tag{36}$$

We used the notation $\epsilon(\sigma; x_1, \dots, x_n)$, the Koszul sign given by the permutation σ of the elements x_i and the convention that $x_1 \otimes \cdots \otimes x_n = x_{\sigma(1)} \dots x_{\sigma(n)} = 1_{\mathbb{K}}$ when $n = 1$. In particular, $\Delta(1_{\mathbb{K}}) = 1_{\mathbb{K}} \otimes 1_{\mathbb{K}}$.

Most people rather work with the *reduced* tensor/symmetric coalgebras:

Example A.3. One defines $\overline{TV} = \oplus_{k=1}^{\infty} V^{\otimes^k}$ and $\overline{SV} = \oplus_{k=1}^{\infty} S^k V$. They are coalgebras for the coproducts (both denoted by $\bar{\Delta}$) given by replacing the element $1_{\mathbb{K}} \in V^{\otimes^0} = S^0 V = \mathbb{K}$ by 0 in eq. (35) and (36). In other words:

$$\Delta(x) = \bar{\Delta}(x) + 1_{\mathbb{K}} \otimes x + x \otimes 1_{\mathbb{K}}.$$

Definition A.4. A *coderivation* of a coalgebra (W, Δ) consists of a linear endomorphism Q of W satisfying the following (co) Leibniz condition:

$$(Q \otimes Id + Id \otimes Q) \circ \Delta = \Delta \circ Q. \tag{37}$$

One denotes by $Coder(W)$ the set of coderivations of (W, Δ). It is a graded Lie subalgebra of $(L(W), [-, -])$.

Remark A.5. If both Q and Q' are odd, then $[Q, Q'] = Q \circ Q' + Q' \circ Q$. This means that if Q is odd, then $Q \circ Q$ is a coderivation.

Definition A.6. A *homological* coderivation consists in a degree one coderivation Q satisfying

$$Q \circ Q = 0. \tag{38}$$

From now on we work with non-negatively graded coalgebras, i.e. such that $W = \oplus_{i \geq 0} W_i$. Let Q be a linear endomorphism of W. As a linear map, it is uniquely defined by its restrictions to the subspaces W_k: if one denotes $Q_k := Q|_{W_k}$, one has $Q = \prod_{k=0}^{\infty} Q_k$. Let us consider the natural projection $\Pi_{W_l} \colon W \to W_l$, for every l. One denotes $Q^l := \Pi_{W_l} \circ Q$. Clearly:

$$Q = \prod_{k,l=0}^{\infty} Q_k^l.$$

Definition A.7. The collection $Q^1 := \{Q_0^1, \ldots, Q_i^1, \ldots\}$ is called the set of *Taylor coefficients* of Q. The coderivation Q is said to be *quadratic* if its only non zero Taylor coefficient is Q_2^1.

Coderivations are most of the time encountered through their Taylor coefficients. Proposition A.8 shows how to reconstruct a coderivation from its Taylor coefficients, and formula (40) expresses the condition of being homological in these terms:

Proposition A.8. *A coderivation Q of $\overline{TV}$ (resp. $\overline{SV}$) is uniquely determined by the collection $\{Q_1^1, \ldots, Q_i^1, \ldots\}$ of its Taylor coefficients by the formula*

$$Q_n^i = \sum_{s=1}^{i} Id^{s-1} \otimes Q_{n-i+1}^1 \otimes Id^{i-s},$$

resp.

$$Q = m_{\overline{SV}} \circ (Q^1 \otimes Id) \circ \bar{\Delta}, \tag{39}$$

where $m_{\overline{SV}}$ denotes the multiplication of $\overline{SV}$.

Prop. A.8, whose proof can be found in [1], [5] or the appendix of [26], enables to reformulate the condition (38) for a coderivation to be homological:

Lemma A.9. *A coderivation Q of $\overline{TV}$ is homological if and only if its Taylor coefficients satisfy the set of equations ($n \geq 1$)*

$$\sum_{i=1}^{n} \sum_{s=1}^{i} Q_i^1 \circ (Id^{s-1} \otimes Q_{n-i+1}^1 \otimes Id^{i-s}) = 0. \tag{40}$$

In particular, a quadratic homological coderivation of $\overline{TV}$ is equivalent to the equation

$$Q_2^1 \circ (Q_2^1 \otimes Id) + Q_2^1 \circ (Id \otimes Q_2^1) = 0. \tag{41}$$

In the same way, a coderivation Q of $\overline{SV}$ is homological if and only if its Taylor coefficients form a $L_\infty[1]$-algebra on V (see Def. 1.1).

Proof. Let Q be a homological coderivation of $\overline{TV}$. By Remark A.5, $Q \circ Q$ is a coderivation, and we can apply Proposition A.8. Therefore we will get a series of equations, namely, the annihilation of all its Taylor coefficients $(Q \circ Q)^1_i$. But one has, by use of Proposition A.8, the following expression for the these coefficients:

$$\begin{aligned} (Q \circ Q)^1_n &= \sum_{i=1}^{n} Q^1_i \circ Q^i_n \\ &\overset{Prop\ A.8}{=} \sum_{i=1}^{n} Q^1_i \circ (\sum_{s=1}^{i} Id^{s-1} \otimes Q^1_{n-i+1} \otimes Id^{i-s}). \end{aligned}$$

The proof of the statement for $\overline{SV}$ goes along the same lines and can be found in [1]. □

Lemma A.9 is important, since it is the link between homological coderivations and algebras. But to have this link explicit, one still needs to "desuspend" the relation. First, let us recall the shift operator $[1]\colon V \to V[1]$, which maps an element of $v \in V_i$ to itself seen as an element of $(V[1])_{i-1}$. (In other words, $[1]$ shifts the degree of an element by 1.) Sometimes we write $v[1]$ for $[1]v$.

Definition A.10. Let Q be a coderivation of $\overline{T(V[1])}$. We define the *desuspension operator* d by

$$dQ^1_n := [1]Q^1_n[-1]^{\otimes^n} : \overline{TV} \to V, \tag{42}$$

and similarly for $\overline{S(V[1])}$.

This desuspension operator constitutes the link between homological coderivations and homotopy algebras. This is the content of the following proposition, whose proof can be found in [1] and [5].

Proposition A.11. *The operator d defined by equation (42) gives a bijection between the sets of quadratic homological coderivations of $\overline{T(V[1])}$ and of associative algebra structures on V.*

Similarly, it also gives a bijection between the sets of homological coderivations of $\overline{S(V[1])}$ and of L_∞-algebra structures on V. The latter restricts to a bijection between the quadratic homological coderivations of $\overline{S(V[1])}$ and the graded Lie algebras structures on V.

This last result suggests the definition of an A_∞-algebra, introduced by Stasheff in [32].

Definition A.12. An *associative algebra up to homotopy* (or *A_∞-algebra*) is a graded vector space V equipped with a collection of maps $\{m_1, \dots, m_l, \dots\}$, obtained by desuspension of the Taylor coefficients of a homological coderivation Q of $\overline{T(V[1])}$.

A.3 The proof of Prop. 2.17: infinite dimensional L_∞-algebras via derived brackets

It is well-known that an L_∞-algebra structure on a finite dimensional graded vector space V is equivalent to a homological vector field on $V[1]$. The L_∞-multibrackets can be recovered with a derived bracket construction [33, Ex. 4.1]. If V is an infinite dimensional, the

above procedure does not apply (it involves considering the dual of $V[1]$). Instead, as stated in Lemma A.9, a L_∞-structure on V can be encoded by a suitable *coderivation* on a *reduced* symmetric coalgebra. In this section we show that the L_∞-structure can also be recovered from a coderivation on a (unreduced) symmetric coalgebra by a derived bracket construction, proving Prop. 2.17.

Let W be a (possibly infinite dimensional) graded vector space. We will apply Voronov's derived bracket construction (Thm. 1) to the graded Lie algebra $(Coder(SW), [-,-])$ of Def. A.4. Let us introduce the abelian subalgebra $\mathfrak{a}$ which we will need.

Lemma A.13. *For every homogeneous $w \in W$,*

$$\alpha_w \colon SW \to SW,$$
$$x_1 \dots x_n \mapsto w \cdot x_1 \dots x_n$$

is a coderivation of SW of degree $|w|$. Further, $\mathfrak{a} := \{\alpha_w : w \in W\}$ is an abelian Lie subalgebra of $Coder(SW)$.

Proof. We show that α_w is a coderivation, i.e., that is satisfies eq. (A.4). With the notations $x_\emptyset := 1$ and $x_I := x_{i_1} \cdot \dots \cdot x_{i_n}$ for $I = \{i_1, \dots, i_n\}$, one can abbreviate

$$\Delta x_I = \sum_{I' \amalg I'' = I} \pm x_{I'} \otimes x_{I''},$$

where $\pm$ are the signs which appear in formula (36). On the one hand one has

$$(\alpha_w \otimes Id + Id \otimes \alpha_w)\Delta(x_I) = \sum_{I' \amalg I'' = I} \pm w \cdot x_{I'} \otimes x_{I''} + (-1)^{|w||x_{I'}|} \pm x_{I'} \otimes w \cdot x_{I''}.$$

On the other hand, if one denotes $w \cdot x_I = x_{\{\star \amalg I\}}$, one gets

$$\Delta(\alpha_w(x_I)) = \sum_{J' \amalg J'' = \star \amalg I} \pm x_{J'} \otimes x_{J''} = \sum_{I' \amalg I'' = I} \pm w \cdot x_{I'} \otimes x_{I''} + (-1)^{|w||x_{I'}|} \pm x_{I'} \otimes w \cdot x_{I''}.$$

Hence α_w is a coderivation.

To show that $\mathfrak{a}$ is abelian we compute for all homogeneous $v, w \in W$ and $x \in SW$ that $[\alpha_v, \alpha_w]x = v \cdot w \cdot x - (-1)^{|v||w|} w \cdot v \cdot x = 0$. □

Lemma A.14. *For every $\tau \in Coder(SW)$ one has $\tau(1) \in W$.*

Proof. For any element $w \in SW$ the following holds: $\Delta w = w \otimes 1 + 1 \otimes w$ iff $w \in W$.

Applying eq. (37) to $1 \in SW$ we see that $\tau(1)$ satisfies the above relation, so it must lie in W. □

There is a canonical embedding of the coderivations on $\overline{SW}$ into those on SW:

Lemma A.15. *Consider the map*

$$\mathcal{J} \colon Coder(\overline{SW}) \to Coder(SW)$$

defined by $(\mathcal{J}\Theta)(1) = 0$ and, for all $n \geq 1$, by

$$(\mathcal{J}\Theta)(w_1 \dots w_n) = \Theta(w_1 \dots w_n).$$

$\mathcal{J}$ is well defined, injective, and bracket-preserving.

Proof. We check that $\mathcal{J}\Theta$ lies in $Coder(SW)$. The relation (37) is trivially satisfied on the element 1. Let now x be an element of $\overline{SW}$. We have

$$\begin{aligned}(\mathcal{J}\Theta \otimes Id + Id \otimes \mathcal{J}\Theta)(\Delta(x)) &= \Theta(x) \otimes 1 + 1 \otimes \Theta(x) + (\Theta \otimes Id + Id \otimes \Theta)(\bar{\Delta}(x)) \\ &= \Theta(x) \otimes 1 + 1 \otimes \Theta(x) + \bar{\Delta}(\Theta(x)) \\ &= \Delta(\mathcal{J}\Theta(x)),\end{aligned}$$

where in the first equality we used $(\mathcal{J}\Theta)(1) = 0$ and in the second that $\Theta \in Coder(\overline{SW})$. Hence $\mathcal{J}\Theta$ is a coderivation.

$\mathcal{J}$ is bracket-preserving since, for any $\Theta_i \in Coder(\overline{SW})$, the graded commutator $[\mathcal{J}\Theta_1, \mathcal{J}\Theta_2]$ vanishes on $1 \in SW$ and agrees with $[\Theta_1, \Theta_2]$ on $\oplus_{k=1}^{\infty} S^k W$. ◻

Now we are ready to prove Prop. 2.17, which recovers $L_\infty[1]$-algebra structures on W via derived brackets. We repeat the proposition for the reader's convenience:

Proposition. *Let W be an L_∞[1]-algebra, and Θ the corresponding coderivation of $\overline{SW}$ given by Lem. A.9. The following quadruple forms a V-data:*

- *the graded Lie algebra $L := Coder(SW)$*
- *its abelian subalgebra $\mathfrak{a} := \{\alpha_w : w \in W\}$*
- *the projection $P\colon L \to \mathfrak{a}$, $\tau \mapsto \alpha_{\tau(1)}$*
- *$\Delta := \mathcal{J}\Theta$.*

The induced $L_\infty[1]$-structure on $\mathfrak{a}$ given by Thm. 1 is exactly the original L_∞[1]-structure on W, under the canonical identification $W \cong \mathfrak{a}, w \mapsto \alpha_w$.

Proof. $\mathfrak{a}$ is an abelian Lie subalgebra of $Coder(SW)$ by Lemma A.13. The map P is well-defined by Lemma A.14, and is clearly a projection (that is, $P^2 = P$). Its kernel $ker(P)$ agrees with the subspace of coderivations vanishing on $1 \in SW$. Hence $ker(P)$ is a Lie subalgebra of $Coder(SW)$ and it contains $\mathcal{J}\Theta$. Further $[\mathcal{J}\Theta, \mathcal{J}\Theta] = 0$ by Lemma A.15.

We conclude that $(L, \mathfrak{a}, P, \Delta)$ is a V-data and the assumptions of Thm. 1 are satisfied.

To compute the induced multibrackets on $\mathfrak{a}$, notice that for every $n \geq 1$

$$[\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}](1) = \mathcal{J}\Theta \circ \alpha_{w_1} \circ \cdots \circ \alpha_{w_n}(1) + \sum_{i=1}^{n} \alpha_{v_i} M_i \tag{43}$$

for certain elements $M_i \in \oplus_{k=1}^{\infty} S^k W$. In particular the sum on the r.h.s. lies in $\oplus_{k=2}^{\infty} S^k W$. Hence

$$\begin{aligned}[\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}](1) &= pr_W\big([\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}](1)\big) \\ &= pr_W\big(\mathcal{J}\Theta \circ \alpha_{w_1} \circ \cdots \circ \alpha_{w_n}(1)\big) \\ &= pr_W\big(\mathcal{J}\Theta(w_1 \cdots w_n)\big) \\ &= \{w_1, \cdots, w_n\}\end{aligned}$$

where in the first equality we used Lemma A.14, in the second we used eq. (43), and in the fourth Lemma A.9. Hence

$$P[\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}] = \alpha_{[\cdots[\mathcal{J}\Theta, \alpha_{w_1}], \cdots, \alpha_{w_n}](1)} = \alpha_{\{w_1, \cdots, w_n\}}.$$

◻

A.3.1 Application: L_∞-algebras associated to A_∞ algebras

It is well known ([21] or [22, Prop. 13.2.16]) that, in the same way that one can associate a Lie algebra to an associative algebra (by taking the commutator), one can associate a L_∞-algebra to an A_∞-algebra. In this subsection, which is not used in the rest of the paper, we show that it is indeed possible to understand this in terms of derived brackets.

Let W be a (possibly infinite dimensional) graded vector space and $w \in W$. Let us define the map

$$\alpha_w \colon TW \longmapsto TW, \quad \alpha_w(w_1 \otimes \cdots \otimes w_n) = \sum_{i=0}^{n} w_1 \otimes \cdots \otimes w_i \otimes w \otimes w_{i+1} \otimes \cdots \otimes w_n.$$

In the following we use the notations of the previous section §A.3, modulo the replacement of the symmetric product by the tensor product.

Proposition A.16. *Let $W[-1]$ be an A_∞-algebra, and Θ the corresponding coderivation of $\overline{TW}$ given by Def. A.12. The following quadruple forms a V-data:*

- *the graded Lie algebra $L := Coder(TW)$*
- *its abelian subalgebra $\mathfrak{a} := \{\alpha_w : w \in W\}$*
- *the projection $P \colon L \to \mathfrak{a}$, $\tau \mapsto \alpha_{\tau(1)}$*
- *$\Delta := \mathcal{J}\Theta$.*

The induced $L_\infty[1]$-structure on $\mathfrak{a}$ given by Thm. 1 is exactly the L_∞[1]-structure on W obtained by symmetrization of the $A_\infty[1]$-algebra structure on W.

Proof. One can easily check by mimicking §A.3 that the map α_w is a coderivation of TW, $\mathfrak{a}$ is an abelian subalgebra of L, and $Ker(P)$ is a subalgebra of L. To show that $(L, \mathfrak{a}, P, \Delta)$ forms a V-data, it remains to show that $[\Delta, \Delta] = 0$. But by definition A.12, an A_∞-algebra on $W := V[-1]$ is equivalent to a Maurer-Cartan element of $Coder(\overline{TW})$, i.e. a coderivation Θ of degree 1 such that $[\Theta, \Theta] = 0$. Now use the fact that the map $\mathcal{J}$ is bracket preserving.

Therefore the derived bracket construction of Thm. 1 can be applied to the V-data $(L, \mathfrak{a}, P, \Delta)$ above, associating a $L_\infty[1]$-algebra to the given A_∞-algebra.

It remains to check that the obtained $L_\infty[1]$-structure on W can alternatively be obtained by symmetrization: The computation following eq. (43) gives in particular

$$pr_W\big(\mathcal{J}\Theta \circ \alpha_{w_1} \circ \cdots \circ \alpha_{w_n}(1)\big) = \{w_1, \cdots, w_n\}. \tag{44}$$

One remarks (proof by induction) that $\alpha_{w_1} \circ \cdots \circ \alpha_{w_n}(1) = \sum_{\sigma \in S_n} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}$. So (44) rewrites as

$$\{w_1, \cdots, w_n\} = \sum_{\sigma \in S_n} \Theta^1_n(w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}),$$

i.e. the n-th bracket of the $L_\infty[1]$-structure on $\mathfrak{a}$ given by Thm. 1 is obtained by symmetrization of the n-th bracket of the original $A_\infty[1]$-structure. $\square$

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